LECTURE 2: ELIMINATION

$$\begin{aligned} x + 2y + z &= 2 \\ 3x + 8y + z &= 12 \\ 4y + z &= 2 \end{aligned} \qquad Ax = b$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

1) ELIMINATION $\rightarrow$ SUCESS OR FAILURE
2) BACK SUBSTITUTION
3) ELIMINATION MATRICES
4) MATRIX MULTIPLICATION

1) WE TRY TO REMOVE A VARIABLE

$$\text{1ST PIVOT} \rightarrow \begin{matrix} \boxed{1} & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{matrix} \quad \longrightarrow \quad \begin{matrix} 1 & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 4 & 1 \end{matrix}$$

2ND PIVOT

WE GOT A ZERO IN [2, 1]

NEXT STEP WOULD BE [3, 1] BUT IS ALREADY 0

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & \boxed{5} \end{bmatrix}$$

U (UPPER TRIANGULAR)

3RD PIVOT

PURPOSE OF ELIMINATION IS TO GET FROM $A \rightarrow U$

PIVOTS CAN'T BE ZERO

IF PIVOT POSITIONS ARE ZERO'D CAN SHUFFLE ROWS

$$\begin{matrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ 0 & 4 & 1 \end{matrix} \quad \longrightarrow \quad \begin{matrix} 1 & 2 & 1 \\ 0 & 0 & -2 \\ 0 & 4 & 1 \end{matrix}$$

FAILURE!

$$\begin{matrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & -4 \end{matrix} \Rightarrow \Rightarrow \begin{matrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & \boxed{0} \end{matrix}$$

← NO THIRD PIVOT

SO NON-INVERTIBLE MATRICES CAN'T BE ELIMINATED

2) BACK SUBSTITUTION. NOW WE BRING IN B

$$\begin{array}{cc} A & b \\ \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} & \begin{bmatrix} 2 \\ 12 \\ 2 \end{bmatrix} \end{array} \Rightarrow \begin{array}{cc} A & b \\ \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} & \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} \end{array}$$

$$\Downarrow$$

$$\begin{array}{cc} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix} & \begin{bmatrix} 2 \\ 6 \\ -10 \end{bmatrix} \\ U & c \end{array}$$

c is b when we get a to U

$$x + 2y + z = 2 \qquad z = -2$$

$$2y - 2z = 6 \qquad y = 1$$

$$5z = -2 \qquad x = 2$$

BACK SUB IS BASIC ALGEBRA

3) MATRIX ELIMINATION

WHAT MATRIX OPERATIONS ARE WE USING?

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

COLUMN MATRIX MULTIPLICATION DOESN'T HELP WITH ROW OPERATIONS

$$\underset{1 \times 3}{\begin{bmatrix} 1 & 2 & 7 \end{bmatrix}} \underset{3 \times 3}{\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}} = \begin{matrix} 1 \times \text{row } 1 \\ 2 \times \text{row } 2 \\ 7 \times \text{row } 3 \end{matrix}$$

MATRIX × COL = COL

ROW × MATRIX = ROW

E_{NAME} TELLS YOU WHERE MATRIX DEVIATES FROM IDENTITY

SO

$$E_{21} \quad \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

$$E_{32} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

COMBINE INTO ONE PROCESS!

$$E_{32}(E_{21} A) = U$$

MOST IMPORTANT FACT ABOUT MTX MULT.

$$(E_{32} \cdot E_{21}) A = U$$

CAN CHANGE THE ORDER OF OPERATIONS
(ASSOCIATIVE LAW)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A = U$$

$\boxed{\text{DIGRESSION}}$

PERMUTATION MATRIX LETS YOU EXCHANGE ROWS

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

SWAPS R_1 w/ R_2

FOR COLS

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

???

LEFT IS ROW, RIGHT IS COLUMNS

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

$a \cdot b \mathrel{!}= b \cdot a$ but $a(b\ c) = (a\ b)c$

$E_{32} \cdot E_{21} =$ ANNOYING?

$A \Rightarrow U$ vs $U \Rightarrow A$. INVERSION!

INVERSES

$$\overset{E^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{E}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \overset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

LECTURE 3

- 4 WAYS OF MULTIPLYING MATRICES
- INVERSES & HOW TO FIND THEM

MULT

$$\underset{A}{i\begin{bmatrix} a_{31} \text{———} \end{bmatrix}} \underset{B}{\overset{j}{\begin{bmatrix} b_{14} \\ | \end{bmatrix}}} = \underset{C = AB}{\begin{bmatrix} c_{ij} \end{bmatrix}} \qquad i,j = \overset{R}{3}, \overset{C}{4}$$

$c_{34} =$ (row 3 of A) $\cdot$ (col 4 of B)

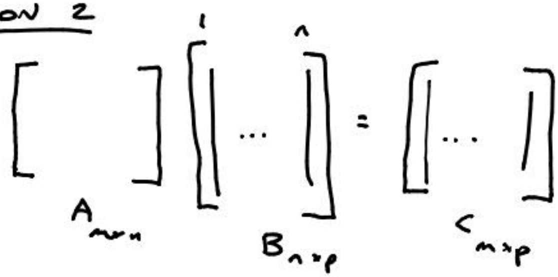
$= a_{31} b_{14} + a_{32} b_{24} + a_{33} b_{34} + \dots$

$= \sum_{k=1}^{n} a_{3k} b_{k4}$ (so n needs to be the same! when cols in a same length as rows in b)

SHAPES AFTER MULT

$A\ m \times n$ $\quad B\ n \times p$ $\quad C = m \times p$

OPTION 2

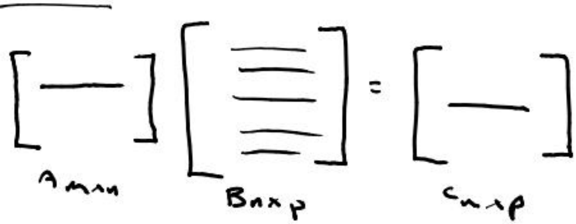


$A \times B\ col_1 = C_{col_1}$ ... etc

column approach to matrix mult

columns of C are combinations of columns of A

OPTION 3



rows of C are combinations of rows of B

OPTION 4

column of A · row of B => $m \times p$ matrix

$m \times 1$ $\qquad$ $1 \times p$

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 6 \end{bmatrix} \qquad \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$$ ← cols are combinations of cols A

rows are " of rows B

4^{TH} WAY: $AB =$ sum of cols of $A \cdot$ rows of B

$$\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 3 & 16 \\ 4 & 24 \end{bmatrix}$$

THE ROW SPACE IS THE LINE THROUGH THE VECTOR $\begin{bmatrix} 1 & 6 \end{bmatrix}$. THE COL SPACE IS THE LINE THROUGH THE VECTOR $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

YOU CAN ALSO CUT UP THE MATRIX AND MULTIPLY VIA BLOCKS

$A_1 B_1 + A_2 B_3$

$$\underset{A}{\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right]} \underset{B}{\left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right]} = \left[\begin{array}{c|c} & \\ \hline & \end{array}\right]$$

(5^{TH} WAY) DERIVATION OF THIS ONE LOOKS FUN

INVERSES (SQUARE FIRST)

A may or may not be invertible

$A^{-1} A$ = Identity matrix

if A^{-1} exists (invertible, non-singular)

square matrices can also do $AA^{-1} = I$

rect matrices can't do this, shapes don't allow it

SINGULAR CASE

$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ not invertible b/c second row not independant of first so combinations can't retrieve I

row/column space doesn't reach identity matrix!

A square matrix has no inverse if we can find non zero x with $Ax = 0$

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} 1 \cdot 3 + 3 \cdot -1 = 0 \\ 2 \cdot 3 + 6 \cdot -1 = 0 \end{matrix}$$

if inverse existed, $x = 0$ by $A^{-1}Ax = 0 \cdot A^{-1}$, but x is not zero, so no inverse exists

INVERTING

$$\underset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}} \underset{A^{-1}}{\begin{bmatrix} a & c \\ b & d \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

solving n systems! $A \cdot j$ of A_{inv} = column j of I

GAUSS-JORDAN (SOLVE 2 EQUATIONS AT ONCE)

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]$$

$\downarrow$ ELIMINATION!

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$\downarrow$ ELIM up!

$$\left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$\uparrow A^{-1}$

GAUSS JORDAN TURNS

$[A \mid I] \Rightarrow [I \mid A^{-1}]$ THROUGH ELIMINATION

WHY DOES THIS WORK?

$EA = I$; $E = A^{-1}$

$$\underline{E}\,[A \quad I] = [I \quad A^{-1}]$$

ELIMINATION MATRICES! $(E_1 \cdot E_2 \cdot E_3 \ldots)$

LECTURE 4

$$A^{-1} \cdot A = I = A A^{-1}$$

AB ___

WE'RE GOING TO FINISH GAUSSIAN ELIMINATION PROPERLY

$AB B^{-1} A^{-1} = I$ ← OBVIOUS WE CAN DO THE MIDDLE ONES FIRST

$B^{-1} A^{-1} A B = I$ ← WHY ARE WE SEEING THIS?

$AA^{-1} = I$

$(A^{-1})^T A^T = I$ ← SO INVERSE OF TRANSPOSE IS TRANSPOSE OF THE INVERSE. WHAT IS THIS TELLING US ABOUT GEOMETRIC STRUCTURE?

ELIMINATION $A = LU$ IS THE MOST BASIC FACTORISATION OF A MATRIX. BUT IS THE RIGHT ENTRY POINT

$$\overset{E_{21}}{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \overset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}$$

$$\overset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \overset{L}{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}} \qquad (L \text{ is } E^{-1})$$

$$= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$$

PIVOTS (JUST AN ALTERNATE VIEW)

3×3 case is more impressive

$E_{32} E_{31} E_{21} A = U$ (NO ROW EXCHANGE)

$A = \underline{\qquad} \, U$

$= E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} U$

$= LU$

WHY IS L BETTER THAN E?

$$\overset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}} \underset{I}{\overset{E_{31}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}} \overset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\overset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}} \overset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \overset{E_{32}E_{21}\ (E)}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{bmatrix}}$$

↑ DON'T LIKE THIS GUY

$EA = U$

INVERSE!

$$\overset{E_{21}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{E_{32}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}} = \overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}}$$

$A = LU$

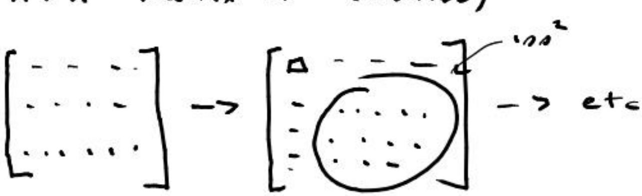
MUCH CLEANER AND NO ROW INTERFERENCE. WHY?

$\boxed{A = LU}$

IF NO ROW EXCHANGES THE MULTIPLIERS GO DIRECTLY INTO L

HOW EXPENSIVE IS ELIMINATION?

$n \times n$ matrix A (100×100)



SO COUNT IS

$$n^2 + (n-1)^2 \ldots 1^2 \quad \sim n^3 \text{ UPPER BOUND}$$

$$\approx \tfrac{1}{3} n^3 \qquad \int_1^n n^2$$

RIGHT HAND SIDE B IS n^2

FACTORISE THE MATRIX AND YOU GET COMPUTATIONAL SHORTCUT LATER

WHAT HAPPENS IF YOU NEED P?

PERMUTATIONS 3×3 (6 of them)

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

EVERY MULTIPLICATION OF THESE IS ALSO ON THE LIST! $P^{-1} = P^T$

PROBLEM SET

1. SUPPOSE YOU HAVE A 3x3 MATRIX A WHERE ROW 3 IS EXACTLY EQUAL TO ROW 1 PLUS ROW 2 (i.e. the rows are linearly dependant).

   A) When you perform elimination to find A=LU, what happens at the bottom of U? $\boxed{\text{ZERO IN THE PIVOT}}$ ✓

   B) can a matrix with this structural dependancy ever be broken down into a valid A=LU factorisation? $\boxed{\text{NO}}$ ✓

2. DESIGN A SINGE 3x3 MATRIX M THAT PERFORMS THE FOLLOWING THREE SPATIAL OPERATIONS SIMULTANEOUSLY WHEN IT MULTIPLIES A VECTOR V = x, y, z

   swaps x and z

   leaves y alone

   streches new z by factor of 5

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} i \\ j \\ k \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} k \\ j \\ i \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 5 & 0 & 0 \end{bmatrix} = \begin{bmatrix} k \\ j \\ 5i \end{bmatrix} \quad \checkmark$$

solution to squared distance X vs v is

```
dist = np.sum((X-v)**2, axis=1)
```

LETS FORMALISE THE IMPLEMENTATION

X is a $n \times 3$ matrix

$$\left[\begin{array}{ccc} x_1 & y_1 & z_1 \\ \hline x_n & y_n & z_n \end{array}\right]$$

v is a 1×3 vector $\begin{bmatrix} x_v \\ y_v \\ z_v \end{bmatrix}$

we cannot add and subtract in current shape, because we need $(n \times 3) - (n \times 3)$ shapes to make any sense. solution is broadcasting v into $n \times 3$. transpose it (no info loss) and repeat down the rows $\left[\begin{array}{ccc} x_v & y_v & z_v \\ \hline \hline \end{array}\right] = v'$

now we can subtract!

then we square with an element-wise operation which is operating inside the $X - v'$ matrix, not on it. so $(X - v')**2$ is also an $n \times 3$ matrix

$$\left[\begin{array}{ccc} (x_1 - x_v)^2 & (y_1 - y_v)^2 & (z_1 - z_v)^2 \\ \hline (x_n - x_v)^2 & (y_n - y_v)^2 & (z_1 - z_v)^2 \end{array}\right]$$

numpy will then allow us to collapse the $n \times 3$ matrix into an $n \times 1$ array with np.sum

we specify axis = 1 to sum the rows

so we've gone from

$$\underset{n \times 3}{\begin{bmatrix} & \\ & \end{bmatrix}} \cdot \underset{3 \times 1}{\begin{bmatrix} \\ \end{bmatrix}} \quad \text{to} \quad \underset{n}{\begin{bmatrix} \\ \\ \end{bmatrix}}$$

$\boxed{\text{LECTURE 5}}$

PA = LU

VECTOR SPACES AND SUBSPACES

— THE BEGINNING OF LINEAR ALGEBRA!

PERMUTATIONS AGAIN

P: EXECUTE ROW EXCHANGES TO GET GOOD PIVOTS

COULD NEED ONE FOR EACH ROW!

$IA = LU =$ 

THIS IS GOING TO WORK BECAUSE P IS CLOSED GROUP AND $P^{-1} = P^T$

COMPUTATIONALLY WE LOOK FOR BIGGEST PIVOT #!

$P^{-1} = P^T$, CLOSED GROUP

$PA = LU$ // any non-singular A

mostly P is I :>

P group size is $n!$ for $n \times n$ space

$A^T A = I$ is not just true for permutations

. . .

what's a transpose? we know this

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix} = R^T$$

TRANSPOSE GENERAL DEF

$$(A^T)_{ij} = A_{ji}$$

SYMMETRIC MATRICES: $A^T = A$ (I is obv example)

(JUST POINTING THIS OUT SO IT'S IN OUR HEADS)

GO BACK TO $R \rightarrow R^T R$ IS ALWAYS SYMMETRIC (WHICH IS WHY IT'S IN PSEUDO-INVERSE)

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 11 & 7 \\ 11 & - & - \\ 7 & - & - \end{bmatrix}$$

WHY? $(R^T R)^T$

WHEN YOU TRANSPOSE THE ORDER IS REVERSED

SO $(R^T R)^T = R^T R^{TT} = R^T R \quad \therefore$ SYMMETRICAL

LET'S GET TO THE GOOD SHIT!!

WHAT ARE VECTOR SPACES?

WHAT ARE SUBSPACES?

WHAT DO WE DO WITH VECTORS?

- ADD TO OTHER VECTORS (COMBINE)
- MULTIPLY BY SCALAR (STRETCH)

LOTS OF VECTORS THAT OBEY COMBINATION & STRETCHIN

VECTOR SPACES

EXAMPLES! R^2 = all real 2-dimensional vectors $\begin{bmatrix}3\\2\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}\pi\\e\end{bmatrix}$

R^2 is the "xy plane" etc

it's a vector space because it's complete

R^3 = all vectors w/ 3 real components $\begin{bmatrix}3\\2\\0\end{bmatrix}$

R^n = all vectors with n real components.

they're columns for now, which ok

SO VECTOR SPACES ARE GROUPS, RIGHT?

SATISFYING THE EIGHT RULES:

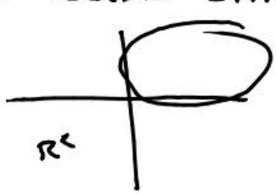
ADDITION

- COMMUTATIVE $u + v = v + u$
- ASSOCIATIVE $u + (v + w) = (u + v) + w$
- ADDITIVE I $u + 0 = u$
- ADDITIVE INVERSE $u + u^{-1} = 0$

MULTIPLICATION

- DISTRIBUTIVE + $a(u+v) = au + av$
- DISTRUBUTIVE × $(a+b)v = av + bv$
- ASSOCIATIVE × $a(bv) = (ab)v$
- MULTIPLICATIVE I $1v = v$

NOT A VECTOR SPACE:



TAKE TOP RIGHT QUADRANT

CAN ADD SAFELY. BUT MULTIPLYING BY NEGATIVE SCALARS DOESN'T WORK. NOT CLOSED UNDER MULTIPLICATION

ARE THERE VECTOR SPACES WITHIN R^n

Straight line $\underbrace{\text{through origin}}_{\text{otherwise can't mult by 0}}$ is a subspace in R^2

subspaces of R^2: (1) R^2

(2) any line through the origin $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

L

not R^1 because it can rotate and vectors have two components

(3) the origin $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Z

subspaces of R^3: (1) R^3

(2) PLANE THROUGH ORIGIN

(3) LINE THROUGH ORIGIN

(4) Z

HOW DO THESE RELATE TO MATRICES?

$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$ columns in R^3

all their linear combinations form a subspace

called THE $\underbrace{\text{COLUMN SPACE}}$ C(A)

$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$ fill out planar subspace through origin

$\boxed{\text{LECTURE 6:}}$ COL SPACES & NULL SPACES

VECTOR SPACES AND SUBSPACES

COLUMN SPACE OF A: solving $Ax = b$

NULL SPACE OF A

VECTOR SPACE REQUIREMENTS

$v + w$ and cv are in the space

all combs $cv + dw$ "

LINEAR COMBINATIONS ARE CLOSED!

SUBSPACES ARE A VECTOR SPACE INSIDE A VECTOR SPACE. PLANE THROUGH ZERO IS SUBPACE OF R^3, eg.

2 SUBSPACES: P and L

$P \cup L$ = all vectors in P or L or both

this ~~is~~ (is not) a subspace

unless the line is in the plane :)

$P \cap$: all vectors in both P & L

this is a subspace because it's just Z

INTERSECTIONS OF SUBSPACES ARE ALWAYS SUBSPACES. $S \cap T$ <u>IS</u> A SUBSPACE

TAKE VECTORS IN THE INTERSECTION AND COMBINATIONS AND STRETCHES WILL BE IN BOTH SUBSPACES AND THEREFOR IN THE INTERSECTION

COLUMN SPACE OF A

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

col space is subspace of R^4

$C(A)$ is all linear combinations of the three columns

$C(A)$ is not R^4 because it can't reach all of it. is 2-dimensional subspace because 3 independant vectors

does $Ax=b$ always have a solution for every b? $\boxed{\text{NO}}$

which right hand sides are ok? the ones in $C(A)$

$\underbrace{Ax = b}$

4 equations, 3 unknowns

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

some vectors b which are not in $C(A)$

WHICH b ALLOW THIS SYSTEM TO BE SOLVED?

obv ANYTHING IN $C(A)$ ← THIS IS WHY THE COLUMN PICTURE IS PREFERRED

Ax = column space for any x, so can solve b exactly if within that space

are the three columns independant for A?

no → col 3 is sum of col 1 and col 2

(and col 1 is a combination of 3 and 2)

NULL SPACE

SAME A: $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$

null space of A contains all solutions $x \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to the equations

$Ax = 0$

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

x IS IN R^3; $C(A)$ IN R^4

NORMALLY WE ELIMINATE TO FIND BUT

$$A_x = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

NULLSPACE CONTAINS Z; PROPERLY $c \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

THIS IS A LINE IN R^3

CHECK $A_x = 0$ GIVES A SUBSPACE

IF $A_v = 0$ & $A_w = 0$; $A_{(v+w)} = 0$ ✓

$A_v + A_w = 0$ (DISTRIBUTIVE LAW)

IF $A_v = 0$, $A_{v \cdot n} = 0$; $nA_v = 0$ ✓

LET'S FIND SOLUTIONS

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$x = [1 \; 0 \; 0]$ ← not a subspace; doesn't go through Z

$[0 \; -1 \; 1]$

column space is built up, null space is solved

OK, SO HOW DO WE THINK ABOUT AND USE THE SUBSPACE WE JUST GOT?

LECTURE 7 : SOLVING $Ax = 0$: PIVOT VARIABLES, SPECIAL SOLUTIONS

COMPUTING THE NULLSPACE $Ax = 0$

TURNING DEFINITION INTO ALGORITHM!

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

linear combo of R1 & R2

multiple of C1

$Ax=0$ extended elimination - continue past zero pivot

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

WHEN WE PERFORM ELIMINATION WE DON'T CHANGE NULLSPACE BUT DO CHANGE COLSPACE

$$\rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

first elimination stage

zero pivots; dependent column. move to next col

$$\begin{array}{cccc} \text{PIVOT COL} & \text{FREE COL} & \text{PIVOT COL} & \text{FREE COL} \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array}$$
$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

echelon form - staircase

2 pivots - rank 2

RANK of A is the numbers of valid pivots

$Ux=0$. SOLVE BY BACKSUBSTITUTION

FREE COLUMNS VARIABLES HAVE OUR CHOICE OF VALUE

IF COL 2 AND 4 ARE FREE, x_2 & x_4 ARBITRARY

WORKED UP

$$x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

WE PICK $x_2 = 1$ and $x_4 = 0$

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$$

$$2x_3 + 4x_4 = 0$$

c·x is a vector in the nullspace. but what is dimension of nullspace? number. of free variables defines. try

$$x = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

WORKED OUT FROM FREE VARIABLES

our two x values are special solutions because we used 1 and 0 for free variables.

$$x = c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

nullspace contains all linear combinations of the special solution

FREE VARS FOR A_{mn} : $N - R$

NEXT STEP: CLEAN UP U

R = reduced row echelon form

$$U = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

ROW OF ZEROS APPEARED BECAUSE r_3 IS COMBO OF ROWS

ELIMINATE UPWARDS! R has zeros above and below pivots

$$\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R = \mathrm{rref}(A)$$

THIS ONLY WORKS BECAUSE SOLUTION IS ZERO, RIGHT?

OTHERWISE YOU'RE STRETCHING..

NOTE I IN PIVOT ROWS & COLUMNS

WHY ARE WE DOING THIS? $Rx = 0$ is easier than $Ux = 0$

$$\begin{array}{cccc} 1 & 0 & 2 & -2 \\ 0 & 1 & 0 & 0 \end{array}$$

PIVOT COLS I FREE COLS F

FOR RREF
FREE COLUMNS TURN UP IN SPECIAL SOLUTION W/ SIGN FLIPPED

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \leftarrow r \text{ pivot rows}$$

r pivot cols, $n-r$ free cols

$$RN = 0$$

$$N = \begin{bmatrix} -F \\ I \end{bmatrix}$$

$$Rx = 0$$

$$\begin{bmatrix} I & F \end{bmatrix} \begin{bmatrix} x_{pivot} \\ x_{free} \end{bmatrix} = 0$$

$$x_{pivot} = -F x_{free}$$

← choose identity

PRACTICE: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \to \overset{u}{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank 2 again

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{rref} \qquad x = c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

putting this to zero just gives us origin

if $R = n$, nullspace is just origin!

$$C = \begin{bmatrix} -F \\ I \end{bmatrix}$$

PRACTICE

$C_1 \cdot -2$

$$\begin{array}{c c} & A = \begin{bmatrix} 1 & -2 & 1 & 3 \\ 2 & -4 & 3 & 8 \\ 3 & -6 & 4 & 11 \end{bmatrix} \\ R_1 + R_2 \longrightarrow & \end{array} \rightarrow \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$\downarrow$$

$$R = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R: \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \& \quad \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

LECTURE 8: $Ax=b$ COMPLETELY SOLVE LINEAR EQUATIONS!

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

we know from $r_1 + r_2 = r_3$ that $b_1 + b_2 = b_3$

$$\begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{bmatrix} = \text{AUGMENTED MATRIX } \begin{bmatrix} A & b \end{bmatrix}$$

Pick b that zeros out bottom row!

$$\begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - 3b_1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix}$$

for $b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$

$0 = b_3 - b_2 - b_1$ ← condition for solvability

what are the conditions on b that make equations solvable?

SOLVABILITY

$Ax = b$ is solvable if and only if b is within colspace of A $C(A)$

if a combination of the rows of A gives the zero row, same combination of B components must be zero

these two statements are equivalent! how?
(come back to this!)

SOLVE $\underline{Ax = b}$ we start w/ one particular solution

① $x_{particular}$: set all free variables to zero, solve $Ax = b$ for pivots

$x_2 = 0$; $x_4 = 0$

$x_1 + 2x_3 = 1 \qquad x_1 = -2$

$2x_3 = 3 \qquad x_3 = 1.5$

$$x_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$$

plug into original to check!

$-2 + 0 \quad 3 \quad 0 : \quad 1$ ✓

$-4 \quad 0 \quad 9 \quad 0 \quad 5$ ✓

$-6 \quad 0 \quad 12. \quad 0 \quad 6$ ✓

(2) NULL SPACE x_{null}

(3) COMPLETE SOLUTION IS $x_{part} + x_{null}$

x_p is the intercept, x_n is the scaffold

$$\left.\begin{array}{l} Ax_p = b \\ Ax_n = 0 \end{array}\right\} A(x_p + x_n) = b$$

zero is magic!

$$\therefore \quad x_c = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

the nullspace tells you how many solutions there are because it's degrees of freedom

subspace affine: shifted from origin

PLOT ALL SOLUTIONS x ← 2D PLANE IN 4D. NO :)

ELIMINATE w/ AUGMENTED MATRIX, FIND x_{part} & x_{null}

QUESTIONS: m by n matrix A of rank r

$r \le m$; $r \le n$

FULL COLUMN RANK: $r = n$ ← nullspace is 0, solution unique if exists

" ROW " : $m = n$

FULL COL RANK: TALL & THIN $\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix}$ rank 2, $n=2$

$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ & F & \end{bmatrix} \leftarrow$

$x_p = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

FULL ROW $r = m$

can solve $Ax = b$ for every b

$\therefore$ solution exists, $n-m$ free variables

$$A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & \\ 0 & 1 & F \end{bmatrix}$$

FULL COL RANK AND FULL ROW RANK

$\downarrow$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ invertible!

$R = I \qquad A(\text{null}) = [0]$

solve by JORDAN-GAUSS TO GET UNIQUE SOLUTION

$r = m = n$

$R = I$ 1 solution to $Ax = b$

$r = n < m$

$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$ 0 or 1 solution to $Ax = b$

$r = m < n$

$R = \begin{bmatrix} I & F \end{bmatrix}$ $m - n$ dimensional solutions to $Ax = b$

could be mixed in

$r < m$ & $r < n$

$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ 0 or infinite solutions

THE RANK TELLS YOU EVERYTHING ABOUT # SOLUTIONS

PRACTICE:

$$A = \begin{bmatrix} 1 & -2 & 1 & 3 \\ 2 & -4 & 3 & 8 \\ 3 & -6 & 4 & 11 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$$

1. $b_3 - b_2 - b_1 = 0$ ✓

2. $Ab \leftarrow$ ELIMINATION

$$\begin{bmatrix} 1 & -2 & 1 & 3 & b_1 \\ 2 & -4 & 3 & 8 & b_2 \\ 3 & -6 & 4 & 11 & b_3 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & 1 & 3 & b_1 \\ 0 & 0 & 1 & 2 & b_2 - 2b_1 \\ 0 & 0 & 1 & 2 & b_3 - 3b_1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & 1 & 3 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_2 = 0, \; x_4 = 0 \qquad x_3 = 1 \qquad x_1 = 1$

$$x_p = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} 1 \\ 0 \\ 4 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

CONSTRUCT A 3×4 MATRIX A THAT SATISFIES THESE CONSTRAINTS:

$r = 2$

$$N(A) = c_1 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

pivot cols are c_1 and c_2

$$R = \begin{bmatrix} 1 & -3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & -3 & 0 & 1 \\ -1 & 3 & 1 & 1 \\ 2 & -6 & 0 & 2 \end{bmatrix}$$

$$Ab = \begin{bmatrix} 1 & 3 & 1 & 2 & b_1 \\ 2 & 6 & 4 & 8 & b_2 \\ 0 & 0 & 2 & 4 & b_3 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 3 & 1 & 2 & b_1 \\ 0 & 0 & 1 & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 3 & 1 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 + 2b_1 \end{bmatrix}$$

valid: $2 - 4 + 2 = 0$

$$x_p = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & 6 & -2 & 3 \\ 2 & 4 & -1 & 3 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 7 \\ 5 \end{bmatrix}$$

$$Ab = \begin{bmatrix} 1 & 2 & -1 & 0 & b_1 \\ 3 & 6 & -2 & 3 & b_2 \\ 2 & 4 & -1 & 3 & b_3 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 2 & -1 & 0 & b_1 \\ 0 & 0 & 1 & 3 & b_2 - 3b_1 \\ 0 & 0 & 1 & 3 & b_3 - 2b_1 \end{bmatrix}$$

$$\Downarrow$$

$$U \quad \begin{bmatrix} 1 & 2 & -1 & 0 & b_1 \\ 0 & 0 & 1 & 3 & b_2 - 3b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 + b_1 \quad \checkmark \end{bmatrix}$$

$$U = \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_p = \begin{bmatrix} 3 & 0 & 1 & 0 \end{bmatrix} \qquad x_n = c_1 \begin{bmatrix} -2 & 1 & 0 & 0 \end{bmatrix} + c_2 \cdot \begin{bmatrix} -3 & 0 & -3 & 1 \end{bmatrix}$$

$$Ab = \left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & b_1 \\ 2 & -2 & 5 & 8 & b_2 \\ 3 & -3 & 7 & 11 & b_3 \end{array}\right] \qquad b = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$\Downarrow$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & b_1 \\ 0 & 0 & 1 & 2 & b_2 - 2b_1 \\ 0 & 0 & 1 & 2 & b_3 - 3b_1 \end{array}\right]$$

$$\Downarrow$$

$$Ub = \left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & b_1 \\ 0 & 0 & 1 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right]$$ b valid!

$$Ub = \left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad x_p = [\underline{-1} \;\; 0 \;\; \underline{1} \;\; 0]$$

$$R = \left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$$Ab = \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & b_1 \\ 2 & 4 & -1 & 8 & b_2 \\ 3 & 6 & -1 & 13 & b_3 \end{array}\right] \qquad b = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

$$\Downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & b_1 \\ 0 & 0 & 1 & 2 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{array}\right]$$

$$\Downarrow$$

$$Ub = \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & b_1 \\ 0 & 0 & 1 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & \underbrace{b_3 - 3b_1 - 2(b_2 - 2b_1)}_{b_3 - 2b_2 + b_1 \;\to\; 8 - 10 + 2 = 0 \;\checkmark} \end{array}\right]$$

$$Ub = \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow R = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 5 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_p = \begin{bmatrix} 3 & 0 & 1 & 0 \end{bmatrix}$$

$$x = x_p + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -5 \\ 0 \\ -2 \\ 0 \end{bmatrix} \quad \checkmark$$

$$Ab = \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & b_1 \\ 2 & 4 & -1 & 8 & b_2 \\ 3 & 6 & -1 & 13 & b_3 \end{array}\right] \qquad b = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

$$\Downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & b_1 \\ 0 & 0 & 1 & 2 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{array}\right]$$

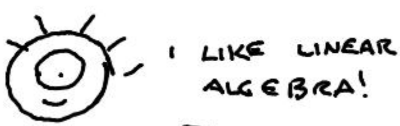
$$\Downarrow$$

$$UB = \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & b_1 \\ 0 & 0 & 1 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{array}\right] \qquad \text{TRY } B = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \checkmark$$

$$RB = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 5 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_p = \begin{bmatrix} 3 & 0 & 1 & 0 \end{bmatrix}$$

$$x_n = s_1 \begin{bmatrix} -2 & 1 & 0 & 0 \end{bmatrix} + s_2 \begin{bmatrix} -5 & 0 & -2 & 1 \end{bmatrix}$$



LECTURE 9: INDEPENDENCE, BASIS & DIMENSION

suppose A is $\underset{\text{rows}}{m}$ by $\underset{\text{cols}}{n}$ with $n > m$ — more unknowns than equations

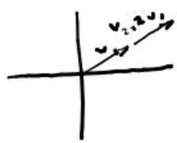
then there are nonzero solutions to $Ax = 0$

reason: there will be free variables!!

Independence

① Vectors $x_1, x_2, \dots, x_n$ are linearly independent if no combination gives the zero vector

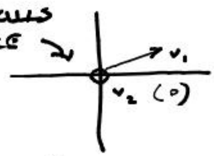
$$c_1 x_1 + c_2 x_2 + \dots c_n x_n \neq 0 \quad \text{(unless all } c = 0\text{)}$$



← DEPENDENT

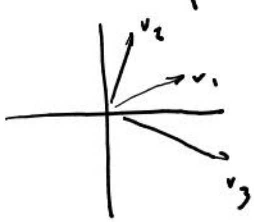
$2v_1 - v_2 = 0$

ZERO VECTOR KILLS INDEPENDENCE



← DEPENDENT

$0v_1 + 1v_2 = 0$



← DEPENDENT BECAUSE OF ①

2D PLANE, 3 VECTORS → $m < n$

$$A = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} - & - & - \\ - & - & - \end{bmatrix} \end{array}$$

IS THIS A RESTATEMENT OF R^2 AS A VECTOR SPACE?

REPEAT WHEN $v_1, \dots, v_n$ ARE COLUMNS OF A

THEY ARE INDEPENDENT IS $\mathcal{N}(A)$ IS $\{0\}$ ← $R = n$

DEPENDENT OTHERWISE! ← $R < n$

SPANNING

Vectors $v_1, \dots, v_\ell$ span a space ← ALL LINEAR COMBINATIONS

means: the space consists of all combinations of those vectors

columns of a matrix span a space

BASIS for a space is a sequence of vectors $v_1, v_2, \dots, v_d$ with two properties:

1) independent
2) span the space

a basis can describe whole space

EXAMPLE: $\mathbb{R}^3$ — basis $\begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$ obviously lots more!

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 5 & 8 \end{bmatrix}$$ ← ANOTHER BASIS

$\mathbb{R}^N$ N vectors give a basis if matrix of vectors is non-singular

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$ ← SPANS A PLANE IN $\mathbb{R}^3$; THESE ARE A BASIS FOR THAT PLANE

THIS IS THE COLSPACE OBV

ALL BASES FOR A GIVEN SPACE HAVE SAME NUMBER OF VECTORS.

↑ DIMENSION

INDEPENDENCE: COMBINATIONS $\neq 0$

SPANNING: vectors describing a space

BASIS: vector building blocks of a given space

DIMENSION: # of vectors in the basis

space is $C(A)$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$ ← spans $C(A)$ but not a basis

$$N(A) = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$ etc

$[1 \; 1 \; 1]$, $[2 \; 1 \; 2]$ > A BASIS

RANK: 2!

RANK OF A IS DIMENSION OF $C(A)$

n - RANK IS DIMENSION OF $N(A)$

MATRICES HAVE RANK; SPACES HAVE DIMENSION

SPECIAL SOLUTIONS ARE BASIS OF NULLSPACE

LECTURE 10: FOUR FUNDAMENTAL SUBSPACES

① COLUMN SPACE $C(A)$

② NULL SPACE $N(A)$

③ ROW SPACE = ALL COMBINATIONS OF COLUMNS OF A^T = $C(A^T)$

④ LEFT NULLSPACE = $N(A^T)$

WHY LEFT? ①

TAKE A as $m \times n$

$N(A)$ is in R^n

$C(A)$ is in R^m

$C(A^T)$ is in R^n

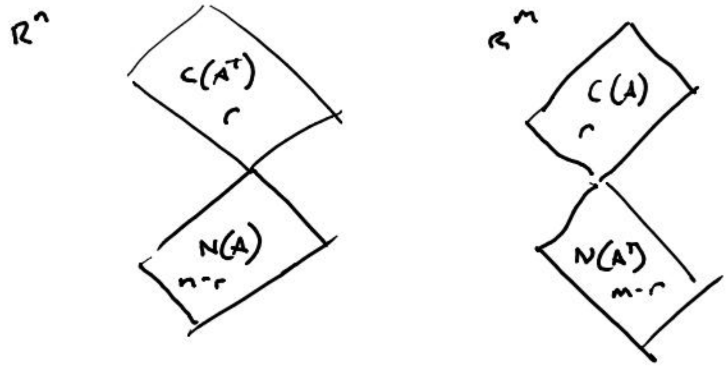
$N(A^T)$ is in R^m

① BECAUSE $A^T y = 0$ $[\ ][\ ] = 0$

$y^T A = 0$

$[\ \ ][\ \ ] = 0$ MULTIPLYING ON THE LEFT!

4 SUBSPACES



LET'S UNDERSTAND THESE SPACES

basis? dimension?

$\dim(C(A)) = r$ & 1 basis is the pivot columns

$\updownarrow$

$\dim(C(A^T)) = r$ (1)

$\dim(N(A)) = n-r$ & 1 basis is special solutions

$\dim(N(A^T)) = m-r$ (2)

① BASIS FOR $C(A^T)$

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\downarrow$$

$$R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C(R) \neq C(A)$

b/c THE OPERATIONS ARE LINEAR COMBOS!

row operations preserve rowspace but not colspace!

BASIS FOR ROWSPACE OF A OR R IS THE FIRST r ROWS OF R

② BASIS FOR $N(A^T)$

THE WORK WE DID TO GET TO R REVEALS $N(A^T)$

$$\text{rref } [A_{m\times n} \; I_{m\times m}] \longrightarrow [R_{m\times n} \; E_{m\times m}]$$

WHAT WE DID

AND BECAUSE IT'S THE LEFT NULLSPACE WE DON'T NEED L

$EA = R$. for invertable $R = I$

then $E = A^{-1}$

$$A = \left[\begin{array}{cccc|ccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{cccc|ccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\Downarrow$$

$$RE = \left[\begin{array}{cccc|ccc} 1 & 0 & 1 & 1 & -1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right]$$

BASIS FOR LEFT NULLSPACE

E

BASIS FOR LEFT NULLSPACE IS ROWS OF E THAT CORRESPOND TO ZERO ROWS IN U OR R

NEW VECTOR SPACE!

$\underline{\underline{M}}$ = ALL 3×3 MATRICES!! ← FITS VECTOR SPACE RULES so why not

$A+B$ & cA – not interested in $A \cdot B$ for now

SUBSPACES: upper triangular
symmetric matrices } ∩ diagonal matrices D

dim $D = 3$ (free numbers in 3×3 diagonals)

WHAT IS DIMENSION OF SYMETRICAL 3×3 MATRICES?

$$\begin{bmatrix} - & - & - \\ & - & - \\ & & - \end{bmatrix} \Rightarrow 6$$

BASIS

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

UPPER TRIANGULAR SIMILAR w/o REFLECTION

SHOW SYMMETRICAL MATRICES FORM A SUBSPACE:

$C = \alpha A + \beta B$ ← LINEAR COMBINATION

$C^T = (\alpha A + \beta B)^T$

$C^T = \alpha A^T + \beta B^T$ so by $A^T = A$ & $B^T = B$, $C^T = C$

and addition preserves symmetry.

scalar multiplication is obvious

GAUSS-JORDAN ELIMINATION PRACTICE

$$A = \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 5 & 4 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -3 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & -1 & -1 \\ 0 & 1 & 0 & 4 & -1 & -2 \\ 0 & 0 & 1 & -3 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 1 & 3 \\ 0 & 1 & 0 & 4 & -1 & -2 \\ 0 & 0 & 1 & -3 & 1 & 1 \end{array}\right]$$

LECTURE 11: MATRIX SPACES; RANK 1; SMALL WORLD GRAPHS

WE WERE IN $\overset{①}{M}$ of all 3x3 MATRICES

- you showed $\overset{②}{\text{symmetricals}}$ were a subspace
- upper $\overset{③}{\text{triangular}}$ more diffcult to show b/c - definition intuitively obvious

BASIS FOR ①

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ 9-dimensional

BASIS FOR ②

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ 6-dimensional

② $\cap$ ③

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ 3-dimensional

basis for space is not neccesarily basis for subspace

$$\underbrace{(2) + (3)}_{\text{not union}} = \text{any element of } (2) + \text{any element of } (3)$$

$$= (1)$$

$$\dim(2) + \dim(3) = \dim((2) \cap (3)) + \dim((2) + (3))$$

Another vector space example

$$\frac{d^2 y}{dx^2} + y = 0$$

$$y = \underbrace{c_1 \cos x + c_2 \sin x}_{\text{nullspace/solution space}}$$

this is a vector space!

a basis: $\cos x$, $\sin x$ dimension 2

linear differential equations require you to find a basis for the nullspace

why is this a vector space? (you know)

$\boxed{\text{RANK}}$

what do we know about r?

lets look at $r = 1$

$$A_{2\times 3} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix} \qquad \text{rank } 1$$

rowspace basis: $\begin{bmatrix} 1 & 4 & 5 \end{bmatrix}$

colspace basis: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \end{bmatrix}$$

EVERY RANK 1 MATRIX HAS THE FORM

$$A = \underset{\text{row}}{u}\,\underset{\text{col}}{v^T}$$

IF WE TAKE ANY MATRIX WE CAN BREAK IT DOWN IS A COMBINATION OF $r=1$ MATRICES (BUT FOR LATER)

M = all 5×17 matrices

subset of $r=4$ matrices ← is this a subspace? $\boxed{\text{NO}}$

is the sum of two $r=4$ matrices $r=4$?

no guarantee!

In R^4 $\quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$ $\qquad$ $S =$ all v in R^4 w/ $\sum_{i=1}^{4} v_i = 0$

$$\dim(S) = 3$$

this is the nullspace of $A = [1\ 1\ 1\ 1]$ $(r=1)$

$\dim(N) = \underset{\text{columns}}{n} - \underset{\text{rank}}{r}$

rowspace : $[1\ 1\ 1\ 1]$ dim 1

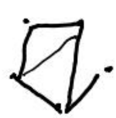
colspace : R^1 dim 1

nullspace : $[-1\ 1\ 0\ 0]$, $[-1\ 0\ 1\ 0]$, $[-1\ 0\ 0\ 1]$ dim 3

left nullspace : $\{0\}$ dim 0

SMALL WORLD GRAPH

what's a graph? = {nodes, edges}



node w/ 5 nodes & 6 edges

(described by 5x6 matrix; for next time)

6-degrees of separation $\rightarrow$ small world

extra notes :

$$\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$$

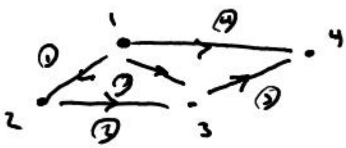
LECTURE 12: GRAPHS, NETWORKS & INCIDENCE MATRICES

APPLICATIONS LECTURE!

REAL LINEAR ALGEBRA USES MATRICES THAT COME FROM SOMETHING

CHEMISTRY DOES ROW REDUCTION ON REACTIONS eg

GRAPH: NODES AND EDGES



$n = 4$ nodes

$m = 5$ edges

we'll work with potential & current today

INCIDENCE MATRIX $A = $

$$\begin{array}{c|l} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} & \begin{array}{l} \text{edge 1} \\ \\ \dots \\ \\ \text{edge 5} \end{array} \\ \text{node 1} \ \dots \ \text{node 4} & \end{array}$$

loop! (edges 1–3)

loop! (edges 3–5)

loops correspond to linearly dependent rows

highly structured; lots of zeros

FUNDAMENTAL SPACES OF A

$N(A) \Rightarrow Ax = 0 \quad \dim N(A) = 1$

$$Ax = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

x = potentials at the nodes

$\downarrow e = Ax$

$x_2 - x_1$, etc.

potential differences across edges

$$x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot c$$ ← nullspace: constant potential

(this is an arbitrary constant)

lets ground node 4 : 0! then we have 3 ind. columns

$r(A) = 3$

$N(A^T) \Rightarrow A^T y = 0 \quad \dim(m-r) = 2$

$$A^T = \begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

KIRCHOFF'S CURRENT LAW

$A^T y = 0$

back to $x_2 - x_1$ etc $\xrightarrow[\text{ohm's law!}]{y = Ce}$ currents of edges

$$A^T = \begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$-y_1 - y_3 - y_4 = 0$

net flow from node 1

conservation law

$y_1 - y_2 = 0$

$y_2 + y_3 - y_5 = 0$

$y_4 + y_5 = 0$

basis for $N(A^T)$

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

← LOOPS! BIG LOOP IS COMBINATION OF LITTLE LOOPS & IS DEPENDENT

$\left(\begin{bmatrix} 1 & 1 & 0 & -1 & 1 \end{bmatrix} \right)$

nullspace of A^T solves kirchoff

rowspace: pivots of A^T

graph w/ no loops

↓

tree!

$$\dim N(A^T) = \overset{\#\text{ edges}}{m} - \underset{(n-1)}{r} = \#\text{ of loops}$$

#nodes − #edges − #loops = 1 ← euler's formula euler's formula true of all graphs but is just dimensional fact of linear algebra

$e = Ax$, $y = Ce$, $A^T y = 0$ ← ec equations fall out of graph matrix!

$A^T C A x = f$ the basis of equilibrium!

resistance

remember $A^T A$ is symmetrical

extra note: C is a diagonal matrix corresponding to physical properties

$A^T C A x = f$ does not include node properties :)

LECTURE 13: REVIEW FOR $Ax = b$

I mean hopefully I know this stuff :)

suppose u, v and w are non-zero vectors in R^7 what are possible dimensions of subspace spanned by u, v, w → 1, 2, 3

$$\underset{\substack{\text{rows} \\ \downarrow}}{5} \times \underset{\substack{\text{cols} \\ \swarrow}}{3}$$

U is a 5×3 matrix, $r = 3$

$N(U) = \{0\}$

$B = \begin{bmatrix} U \\ 2U \end{bmatrix} \quad r = 3$

rref $= \begin{bmatrix} U \\ 0 \end{bmatrix}$

$C = \begin{bmatrix} U & U \\ U & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} U & U \\ 0 & -U \end{bmatrix} \Rightarrow \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} \quad r(C) = 6$

dimension of $N(C^T)$

C is 10×6, $m = 10$, $r(C) = 6$, $\dim(N(C^T)) = 4$

$Ax = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} \quad x = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \underbrace{c \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\dim N(A) = 2}$

$\dim R(A) =$

A is 3×3; $m = 3$, $n = 3$, $r = 1$, $\dim R(A) = 1$

rref $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}$

$Ax = b$ can be solved if b is linear combo of

$\alpha \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

(don't forget other cases $r = m$, $r = n$)

TRUE OR FALSE:

IF $N(A) = \{0\}$, IS $N(A^T) = \{0\}$? TRUE

M = 5x5 MATRICES. DO INVERTIBLE MATRICES FORM

SUBSPACE? FALSE (NO ZERO!)

IF $B^2 = 0$, $B = 0$ FALSE

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \leftarrow \text{DANGEROUS MATRIX}$$

A SYSTEM OF N EQUATIONS & N UNKNOWNS IS

solvable if columns are independent = true

$$\begin{array}{c} \\ \text{this is} \rightarrow \\ \text{ignorable!} \end{array} B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \overset{R}{\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

$CD = 0$

$D = 0 \cdot C^{-1} = 0$

if C is invertible

$Bx = 0$ basis for $N(B) \leq R^4$

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

solve $Bx = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\qquad x_p = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ (first col of multiplied)

IF $m = n$, $R = C$? FALSE

do A and $-A$ share subspaces: TRUE

if A and B share " , is $A = cB$: FALSE!

example: A is invertible 6x6; B sim, $A \,!\!= cB$

if A, B same 4 subspaces, what does that tell us?

- same r

row exchange A, which subspaces stay same? row & null

why can $v=\begin{bmatrix}1\\2\\3\end{bmatrix}$ can't be in nullspace & in a row

nullspace and rowspace are orthogonal :)

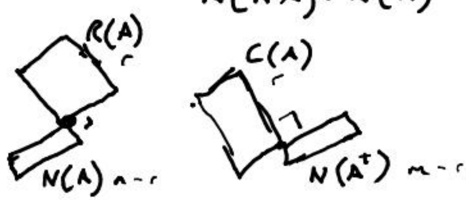
$\dim(R) + \dim(N) = n$; only share zero vector

$R \cap N = \{0\}$

LECTURE 14: ORTHOGONAL VECTORS & SUBSPACES

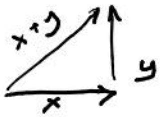
NULLSPACE $\perp$ ROWSPACE

$N(A^TA) = N(A)$



THESE SUBSPACES ARE ORTHOGONAL WITH THEIR PAIR

WHAT DOES IT MEAN FOR VECTORS TO BE ORTHOGONAL? I THINK WE MIGHT SAY THEIR COMPONENTS DON'T INTERFERE WHEN MULTIPLIED



pythagoras

$x^T y = 0$ THIS IS WHAT I SAID BUT CLEANER

$\|x^2\| + \|y^2\| = \|x+y\|^2$ when x & y are orthogonal

connect pythagoras to dot product

what is the length squared of $\{x\}$? $x^T x$

$$\cancel{x^T x} + \cancel{y^T y} = (x+y)^T (x+y)$$

$$= \cancel{x^T x} + x^T y + y^T x + \cancel{y^T y}$$

$$0 = x^T y + y^T x$$

$$= 2x^T y \leftarrow \text{SAME TRICK WE USED IN MATRIX SOLUTION FOR LINEAR REGRESSION!}$$

$$\boxed{0 = x^T y} \ :>$$

EVERY VECTOR PAIRING WITH $\{0\}$ IS ORTHOGONAL

OK, THATS VECTORS. WHAT ABOUT SPACES?

SUBSPACE S IS ORTHOGONAL TO SUBSPACE T ①

① means: every vector in S is orthogonal to every vector in T

- IF SUBSPACES MEET AT NON-ZERO VECTORS, NOT ORTHOGONAL

$S \cap T = \{0\} \leftarrow$ necessary but not sufficient rowspace is orthogonal to nullspace

② colspace " left nullspace

why?

① $Ax = 0$ $\begin{bmatrix} row_1 \\ row_2 \\ \vdots \\ row_n \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \{0\}$

$row_1 A x_1 = 0$; $row_2 A x_2 = 0$; all dots $= 0$, $x^T y = 0$

but now we have to deal w/ all linear combos

$$\left.\begin{aligned} &c_1 (row_1)^T x = 0 \\ + &c_2 (row_2)^T x = 0 \end{aligned}\right\rangle \text{obv}$$

② A^T is just orientation shift

remember $\dim(R(A)) + \dim(R(N)) = n$

$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \end{bmatrix}$

$R(A) = c[1\ 2\ 5]$

$\dim N(A) = 2$ (plane perpendicular to $[1\ 2\ 5]$

$N(A) = c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$

nullspace and rowspace are orthogonal complements w.r.t R^n. nullspace contains ALL vectors perpendicular to rowspace

① "solve"

coming: $Ax = b$ w/ no solutions (projection)

$m > n$ (repeated measurements, big datasets w/ noise)

THROW AWAY EQUATIONS? UNSATISFACTORY

A^TA !!! :> I KNEW IT

square; symmetric

$(A^TA)^T = A^{TT}A^T = A^TA$

① $A^TA\hat{x} = A^Tb$ ← very important

we're hoping $\hat{x}$ has a solution and it's the best one :>

when is A^TA invertible?

$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} = A$ $(m = 3, n = 2)$ $r = 2$

NOTE: $r(A^TA) = r(A)$

$N(A^TA) = N(A)$

this has some implications!

$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 8 & 30 \end{bmatrix}$

A^TA is invertible <u>iff</u> A has independent columns

SIDE QUEST: TAYLOR SERIES

WE WANT TO APPROXIMATE $f(x)$ WITH A POLYNOMIAL $P(x)$

$f(x)$ @ $x = a$

$P(a) = f(a)$ THIS IS JUST MATCHING A

$P'(a) = f'(a)$ FUNCTION'S LOCAL PROPERTIES.

$P''(a) = f''(a)$ $a = 0$ IS A MCLAURIN SERIES

...

$P^{'\dots'}(a) = f^{'\dots'}(a)$

WHEN WE DO A TAYLOR EXPANSION, WE'LL SEE FACTORIALS EVERYWHERE. WHY? POWER RULE

$d/dx\left[x^n\right] = nx^{n-1}$

LET'S BUILD CUBIC MACLAURIN

$P(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$

OUR GOAL IS TO FIND CONSTANTS $c_0 \rightarrow c_3$

SO WE EVALUATE @ 0 AND KEEP DIFFERENTIATING

$P(0) = c_0 = f(0)$

$P'(x) = c_1 + 2c_2 x + 3c_3 x^2 \Rightarrow P'(0) = c_1 \Rightarrow c_1 = f'(0)$

$P''(x) = (2\cdot1)c_2 + (3\cdot2)\cdot c_3 x \Rightarrow c_2 = f''(0)/2!$

$P'''(x) = (3\cdot2\cdot1)c_3 \Rightarrow c_3 = f'''(0)/3!$

GENERAL CASE

$$f(x) = f(a) + f'(a)(x-a) + f''(a)/2! \cdot (x-a)^2 + f'''(a)/3! \, (x-a)^3 \dots$$

The x-a terms are a horizontal shift; just moving your coordinate system to match the function

Δx is $x-a$!!

$$f(a+\Delta x) = f(a) + f'(a)\Delta x + f''(a)\Delta x^2/2 \leftarrow \text{BASIC MOTION!}$$

WHEN Δx IS SMALL, HIGHER ORDER TERMS GET <u>VERY</u> SMALL $\rightarrow$ $0.01^3 = 0.000001$

EXAMPLES: e^x, $\sin x$, $\ln(x)$, $\cos x$

$\cos x$ @ $a=0$

$f(x) = \cos(x) \Rightarrow f(0) = \cos(0) = 1$

$f'(x) = -\sin(x) \Rightarrow f'(0) = 0$

$f''(x) = -\cos(x) \Rightarrow f''(0) = -1$

$f'''(x) = \sin(x) \Rightarrow f'''(0) = 0$

$f''''(0) = 1$

$$\cos(x) = 1 - 1/2!\,x^2 + 1/4!\,x^4 - 1/6!\,x^6 \dots$$

$\sin x$ @ $a=0$

$f(x) = \sin(x) \Rightarrow f(0) = 0$

$f'(x) = \cos(x) \Rightarrow f'(0) = 1$

$f''(0) = 0$

$f'''(0) = -1$

$$f(x) = 1 - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

e^x @ $a=0$

$f(x) = e^x \Rightarrow f(0) = 1$

$f'(x) = e^x \Rightarrow 1$

$\therefore$

$$f(x) = 1 + x + x^2/2! + x^3/3! + \ldots x^n/n!$$

NOTE: $e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \ldots$

$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)$

$= \cos(x) + i\sin(x)$ ← EULER'S THEOREM

WHEN DOES A TAYLOR SERIES APPROXMATION ANALYTICALLY EQUAL THE FUNCTION?

- RADIUS OF CONVERGENCE ← WHERE IS THE SUM FINITE?

e^x ← r_{con} is infinite $\quad 1/(1-x)$ ← r_{con} between -1 & 1

- REMAINDER TERM GOES TO ZERO

WE CAN HAVE SMOOTH FUNCTIONS THAT AREN'T TRACTABLE: $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

$P(x) = 0 + 0x + 0/2!\, x^2 \ldots = 0$

which is not e^{-1/x^2}!

TAYLOR'S THEOREM

$$f(x) = P_n(x) + \underbrace{R_n(x)}_{\text{error from stopping at } n}$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

$\lim_{n \to \infty} R_n(x) = 0$ means TAYLOR EXPANSION = FUNCTION

① THERE IS A LOT HIDING HERE. "c" IS JUST THE POINT AT WHICH THE ERROR TERM IS CAPTURED BY YOUR NUMERICAL CHOICE.

THIS IS TAUTOLOGICAL, BUT WITH ONE TWIST: YOU KNOW IT IS BOUNDED BY x & a

$$\text{ERROR} \leq \frac{M}{(n+1)!}(x-a)^{n+1}$$

where M is a max value between x and a

MULTIVARIABLE FUNCTIONS

LET'S TAKE A VECTOR $w = [w_1, w_2, \dots, w_s]^T$

NOW LET'S USE THAT AS OUR ANCHOR AND APPLY A SMALL DISTANCE Δw

START

$$E(w + \Delta w) = E(w) + \underset{①}{\nabla(w)^T \Delta w} + \frac{1}{2} \Delta w^T \underset{②}{H(w)} \Delta w + \underset{③}{\dots}$$

① GRADIENT VECTOR $\nabla E(w)$; COLLECTS ALL LOCAL PARTIAL DERIVATIVES

$$\nabla E(w) = \begin{bmatrix} \partial E / \partial w_1 \\ \partial E / \partial w_2 \\ \vdots \end{bmatrix}$$

$\nabla E^T \Delta w$ ← how closely Δw aligns w/ steepest local slope

② QUADRATIC TERM

HESSIAN MATRIX TRACKS SECOND DERIVATIVE W ALL DIRECTIONS.

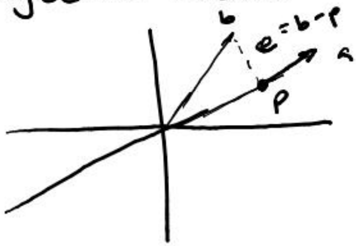
$$H = \begin{bmatrix} \partial^2 E / \partial w_1^2 & \partial^2 E / \partial w_1 \partial w_2 \\ \partial^2 E / \partial w_2 \partial w_1 & \partial^2 E / \partial w_2^2 \end{bmatrix}$$ ← quickly intractable

THIS IS MULTI-DIMENSIONAL CURVATURE

③ HORRIBLE THINGS!

LECTURE 15: PROJECTIONS ONTO SUBSPACES

- projections!
- least squares
- projection matrix



a is 1-d subspace.

where is a closest to b. obv $e \perp a$

- we know p is some multiple x of a
- $a^T(b - xa) = 0$

$x a^T a = a^T b$

$x = \frac{a^T b}{a^T a}$; $p = a x$; ① $p = a \frac{a^T b}{a^T a}$

① THE PROJECTION IS CARRIED OUT BY A MATRIX

proj $p = P \cdot b$ ③

BE CAREFUL NOT TO CANCEL

② $P = \frac{a a^T}{a^T a}$

P is square and colspace P is a

rank P = dim(a)

$P^T = P$ ← key property

$P^2 = P$ ← projecting twice doesn't change result

NOW MOVE TO MORE DIMENSIONS

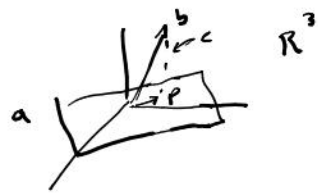
WHY DO WE WANT THIS?

closest solution outside colspace if $ax = b$ has no solutions.

ax in colspce. b not in colspace.

solve $A\hat{x} = P$ instead

best x → projection of b onto colspace



R^3

$\underbrace{\text{plane of } a_1, a_2}_{\text{column space of } A = \begin{bmatrix} a_1 & a_2 \end{bmatrix}}$

$b - A\hat{x} = e \perp A$

$\therefore \underbrace{\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix}}_{A^T} (b - A\hat{x}) = 0$

for plane in R^3. e is in left nullspace

$A^T A \hat{x} = A^T b$

$\hat{x} = (A^T A)^{-1} A^T b$

$P = A\hat{x} = A \underbrace{(A^T A)^{-1}}_{\text{division in 1-d}} A^T b$

$\therefore \; P = A(A^T A)^{-1} A^T$

$P^T = P$

$P^2 = P$??

REMEMBER A IS non-square to make no solution for $Ax = B$, so A^{-1} is meaningless, have to leave $(A^T A)^{-1}$ intact. if $A_{n \times n}$ spans $\mathbb{R}^n$, $P = I$

← drops out

$$P = A(A^TA)^{-1}(A^T \cdot \cancel{A)(A^TA)^{-1}}A^T$$

$\therefore\ P^2 = P$

WHEN DO WE USE?

least squared fitting

e.g. $(1,1), (2,2), (3,2)$ ← BEST LINE THAT GOES THROUGH

$b = C + dt$

$$\left.\begin{aligned} C + D &= 1 \\ C + 2D &= 2 \\ C + 3D &= 2 \end{aligned}\right\} \quad \overset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}} \overset{\hat{x}}{\begin{bmatrix} C \\ D \end{bmatrix}} = \overset{b}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}$$

solve $A^TA\hat{x} = A^Tb$

LECTURE 16: LEAST SQUARES & BEST FIT

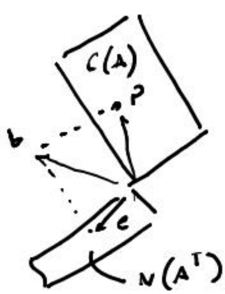
$P = A(A^TA)^{-1}A$

if b in colspace, $Pb = b$ ②

if $b \perp$ " , $Pb = 0$ ①

① b in $N(A^T)$; any multiplication $A^Tb = 0$

② b in $C(A)$; $b = Ax$, things cancel out



$p + e = b$

e is projection onto $N(A^T) = (I - P)b$

BACK TO BEST FIT

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}} \underset{\hat{x}}{\begin{bmatrix} C \\ D \end{bmatrix}} = \underset{b}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}$$

$y = C + Dt$

cols of A are independent

what is best possible solution? minimize squared error $\rightarrow \|Ax - b\|^2 = \|e\|^2$ ← linear regression

don't go to geometry — tedious to do $e_1, \dots, e_n$

LINEAR ALGEBRA PICTURE IS BETTER :)

p_1, p_2, p_3 are points on best line

closest to fitting point. inside column space

find $\hat{x} = \begin{bmatrix} C \\ D \end{bmatrix}$, p

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \end{array}\right] = \left[\begin{array}{cc|c} 3 & 6 & 5 \\ 6 & 14 & 11 \end{array}\right]$$

$A^TA\hat{x} = A^Tb$

$$\left.\begin{array}{l} \partial e/\partial C = 3C + 6D = 5 \\ \partial e/\partial D = 6C + 14D = 11 \end{array}\right\} \text{NORMAL EQUATIONS}$$

we could have used calculus and taken partial derivatives to get normal equations too

solve! $2D = 1$ $D = 1/2$, $C = 2/3$

best line $\rightarrow y = 2/3 + 1/2\, t$

plug back in to e_1, e_2, e_3 and $\rightarrow \overbrace{-1/6, 1/3, -1/6}^{\text{error vector}}$

$b = p + e$

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 7/6 \\ 10/6 \\ 13/6 \end{bmatrix} + \begin{bmatrix} -1/6 \\ 2/6 \\ -1/6 \end{bmatrix}}_{\text{ORTHOGONAL (VERIFY BY MULT)}}$$

$e \perp p$ & $e \perp C(A)$ ← OBVIOUSLY BUT GOOD TO CHECK

BEST FIT IS SOLVING $A^T A \hat{x} = A^T b$

$p = A\hat{x}$

LET'S TALK MORE ABOUT $A^T A$ ← REQUIRES INDY COLUMNS TO INVERT. LET'S PROVE!

Suppose $A^T A x = 0$

prove $x = \{0\}$

we're proving $N(A^T A) = \{0\} \rightarrow$ full rank

TRICK → dot both sides by x^T

$$x^T A^T A x = 0 = (Ax)^T (Ax) \underset{\text{(square)}}{\Rightarrow} Ax = 0 \Rightarrow x = 0$$

A has ind cols

COLUMNS CERTAINLY INDEPENDENT IF $\underbrace{\perp \text{ UNIT } v}_{\text{ORTHONORMAL}}$

e.g. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

LECTURE 17: GRAM-SCHMIDT

ORTHOGONAL VECTORS – USE q TO DENOTE ORTHONORMAL VECTORS

$$q_i \cdot q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

THESE MAKE CALCS NICE!

$$Q = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} \qquad Q^TQ = \underbrace{\begin{bmatrix} q_1^T \\ q_n^T \end{bmatrix}}_{Q^T} \underbrace{\begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}}_{Q} = I$$

THESE AREN'T ORTHOGONAL/ORTHONORMAL MATRICES; WE ONLY USE "ORTHOGONAL" FOR SQUARE MATRICES. IF Q IS SQUARE; $Q^TQ = I$ TELLS US $Q^T = Q^{-1}$.

EXAMPLES: perm $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = I$

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad Q = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \text{①.}$$

① HADAMARD MATRIX – 2, 4, 16, 64

BUT WE DON'T KNOW IF CONSTRUCTABLE FOR n SIZE

$$Q = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$$

DIVIDE BY LENGTH

SUPPOSE Q HAS ORTHONORMAL COLUMNS (DEFINITIONALLY)

PROJECT ONTO ITS COLUMN SPACE

$$P = Q\underbrace{(Q^TQ)^{-1}}Q^T$$

For q_i; $(Q^TQ)^{-1} = I$

$\therefore P = QQ^T = I$ if Q is SQUARE

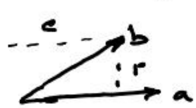
REMEMBER $A^TA\hat{x} = A^Tb$

w/ Q; $Q^TQ\hat{x} = Q^Tb \Rightarrow \hat{x} = Q^Tb$ $(\hat{x}_i = q_i b)$

if basis is orthonormal, projection along the n^{th} basis is q_n^Tb

GRAM-SCHMIDT – WHEN WE DON'T START WITH ORTHONORMAL COLUMN VECTORS

independent vectors a, b



} produce orthogonal vectors A, B $\longrightarrow$ normalise to q_1, q_2: $\frac{A}{\|A\|}$ & $\frac{B}{\|B\|}$

B is going to be b-p right? ✓

$$B = \underset{e_{gram}}{b - A^Tb(A^TA)^{-1}A}$$

$$A^T B = A^T\left(b - \frac{A^T b}{A^T A} A\right) = A^T b - A^T b = 0 \quad \text{shows } B \perp A$$

now $a, b, c \Rightarrow A, B, C \Rightarrow q_1, q_2, q_3$

$$Q = \begin{bmatrix} A & B \end{bmatrix}$$

$$C = c - \underbrace{\frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B}}_{\text{surely a matrix way?}} B$$

EXAMPE

$a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ ← independent, not orthogonal

$$a = A; \quad B = b - \frac{A^T b}{A^T A} A = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$q_1 = A/\|A\| \Rightarrow \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$q_2 = B/\|B\| \Rightarrow \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

how is colspace Q related to colspace A? identical obv

$$Q = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 2 & 2 \end{bmatrix}$$

for elimination $A = LU$

for g-s $A = QR$ (oh thank god)

R is upper triangular ①

$$A = \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix} \begin{bmatrix} & \\ 0 & \end{bmatrix}$$

①

$$R = \begin{bmatrix} a_1^T q_1 & * \\ a_1^T q_2 & * \end{bmatrix}$$

q_2 is orthogonal to a_1; $q_1^T \cdot q_2 = 0$

LECTURE 18: PROPERTIES OF DETERMINANTS

SQUARE MATRICES! EVERY SQUARE MATRIX HAS A DETERMINAT det(A) - magic number!

det(A) = 0 → A is singular (but more)
or
|A|

THREE PRIMARY PROPS of DETERMINANT

① det I = 1

signs are going to be a middlin' pain attention

② permute rows: reverse sign of det

det P = 1 or −1 (depending on # exchanges) for 2×2 .

(1) $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$ $\qquad$ $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

(2) $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

(3) a. $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

b. $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

THE DETERMINANT IS A LINEAR FUNCTION OF ROW IF ALL OTHER ROWS UNCHANGED. $\boxed{\text{LINEAR FOR EACH ROW}}$

WHAT DO THESE PROPERTIES TELL US

I THINK det IS A DESCRIPTION OF ROWSPACE

(4) TWO EQUAL ROWS → det = 0

(USE PROPERTY 2, det sign flips, but det = det; ∴ det = 0)

(5) SUBTRACT $l \times \text{row}_i$ from row_k

det unchanged! → elimination doesn't impact det!

$$\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} \underset{3b}{\longrightarrow} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \underbrace{\begin{vmatrix} a & b \\ -la & -lb \end{vmatrix}}_{\det = 0 \text{ by } 3a,\, 4}$$

(6) row of zeros $\rightarrow$ det $= 0$

(use 3.a w/ $t_1 = 0$; $t_2 = 1$; $0\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1\begin{vmatrix} a & b \\ c & d \end{vmatrix}$; $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$)

(7) $\begin{vmatrix} d_1 & * & * & * \\ & d_2 & * & * \\ 0 & & \ddots & * \\ & & & d_n \end{vmatrix} = (d_1)(d_2)\dots(d_n)$

THIS IS HOW WE CALC DET! IT'S THE PRODUCT OF PIVOTS w/o ROW EXCHANGES; CAREFUL w/ signs

PROVING THIS ONE SEEMS INVOLVED; LEVERAGES (5) TO GET US TO RREF w/ DET UNCHANGED; RREF IS DIAGONAL, THEN GET TO 1 VIA $d_1 d_2 \dots d_n |I|$. CLEVER!

(3a)

(8) a. det $A = 0$ when A is singular

get row of zeros during elim, use (6)

b. det $A \,!= 0$ when A not singular

full pivots; use (7)

now we have formula; look at 2x2

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \rightarrow \begin{vmatrix} a & b \\ 0 & d - \frac{c}{a} b \end{vmatrix} = ad - cb \checkmark$$

asymptote in rowspace?

(9) det AB = det $A \cdot$ det B from (7), trivial

$\therefore$ det $A^{-1} = 1/\det A$ from $A^{-1}A = I$; (1)

$\det A^2 = (\det A)^2$ ← INTERESTING, FROM (3a)

det $2A = 2^n \det A$ (n-dimensional matrix)

(10) $\det A^T = \det A$ switches our rows to columns!

$|A| \to |LU|$ (5)

$|A^T| \to |L^T U^T|$

$|LU| \to |L||U|$ (9)

$|L^T U^T| \to |L^T||U^T|$

$|L| \to 1$ L is triangular w/ diagonals as 1,

$|L^T| \to 1$ (7) gives us $|L|, |L^T| = 1$

$\therefore |U| = d_1, d_2 \ldots d_n = |U^T|$

ONE LAST COMMENT ON DET - IS IT POSSIBLE TO PRODUCE SAME MATRIX w ODD||EVEN ROW EXCHANGES? HOPEFULLY NOT :)

LECTURE 19: DETERMINANT FORMULAS & COFACTORS

- FORMULA FOR $\det A$ ($n!$ TERMS)
- COFACTOR FORMULA
- TRIANGULAR MATRICES

REMEMBER DET IS PRODUCT OF PIVOTS

REMEMBER OUR KEY 3 FACTS

(1) $\det I = 1$ (unit volume) (2) row switches reverse sign

(3) det linear in each row separately

DET FORMULA FOR 2x2

zero det

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \ldots \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}$$

diagonals is here

are

$$\begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} \;\&\; \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix}$$

$ad - bc$ ✓

WHAT ABOUT 3x3?

$$\begin{vmatrix} a & 0 & 0 \\ | & | & | \end{vmatrix} + \begin{vmatrix} 0 & b & 0 \\ | & | & | \end{vmatrix} + \begin{vmatrix} 0 & 0 & c \\ | & | & | \end{vmatrix}$$

... (27 parts; mostly zero)

(WHAT ARE THE NONZEROS?)

$a_{11}a_{22}a_{33}$ $\quad$ $-a_{12}a_{21}a_{33}$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \Rightarrow \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix}, \quad \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix},$$

note col switch

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix},$$

$-a_{11}a_{23}a_{32}$ $\quad$ $a_{12}a_{23}a_{31}$

$$a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$-a_{13}a_{22}a_{31}$$

$n=4$ ← DIFFERENT ROW SWITCH

SURELY WE JUST ELIMINATE AND TAKE DIAGONAL?

BUT GENERALLY:

$$\det A = \sum_{n!\text{ terms}} \pm\, a_{1\alpha} a_{2\beta} a_{3\gamma} \cdots a_{n\omega}$$

rows $1 \ldots n$
cols $\alpha \ldots \omega$

$(\alpha, \beta, \gamma, \omega) =$ permutation of $(1, 2, \ldots, n)$

EXAMPLE $n=4$

$$\begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} = \overset{2\text{ perm}}{1} - \overset{1\text{ perm}}{1} = 0$$

(obvious through elimination)

COFACTORS – BREAK UP THE BIG FORMULA

cofactors in parentheses

START w/ 3×3: $\det = a_{11}(a_{22}a_{33} - a_{23}a_{32})$ ①
$+ a_{12}(\quad \ldots \quad)$ ②
$+ a_{13}(\quad \ldots \quad)$

① det of 2×2 matrix $a_{22} \leftrightarrow a_{33}$

$$\begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix}$$

②
$-a_{12}(a_{21}a_{33} - a_{23}a_{31})$

MATRICES AS OBJECTS :)

COFACTOR OF $a_{ij} \doteq \pm \det\begin{pmatrix} n-1 \text{ matrix} \\ \text{w/ row } i \text{ removed} \\ \text{col } j \end{pmatrix} = C_{ij}$

sign $\rightarrow$ + if $i+j$ is even,
− if $i+j$ is odd

$$\begin{vmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{vmatrix}$$ ← signs of cofactors for 5×5

cofactor formula: $\det A = a_{11}C_{11} + a_{12}C_{12} + \dots a_{1n}C_{1n}$
along row 1

THIS IS OBVIOUSLY RECURSIVE

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$:)

EXAMPLE

$|A_1| = 1 \quad |A_2| = 0 \quad |A_3| = -1$

$$A_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$|A_4| = 1 \cdot A_3 - 1|A_2| = -1$

$|A_5| = 0$

$|A_6| = 1$

$|A_n| = |A_{n-1}| - |A_{n-2}|$

$|A_7| = 1$

periodic! period 6

LECTURE 20: DETERMINANT APPLICATIONS

· FORMULA FOR A^{-1} ← cofactor!

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{for } 2\times 2$$

① $A^{-1} = \dfrac{1}{\det A}\begin{bmatrix} \; C \; \end{bmatrix}^T$ $\quad C$ is a matrix of cofactors

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad C_{11} = ei - fh$$

det A is products of n entries

C is products of n-1 "

WHY DOES ① WORK?

CHECK $\quad AA^{-1} = I \qquad A \cdot \dfrac{1}{\det A} C^T = I$

cofactors of 1st row $\qquad AC^T = \det A \, I$

$$\begin{array}{cc} ② & ③ \\ \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} & \begin{bmatrix} c_{11} & \cdots & c_{n1} \\ c_{12} & & \vdots \\ \vdots & \ddots & \\ c_{1n} & & c_{nn} \end{bmatrix} \end{array} = \begin{bmatrix} \det A & & 0 \\ & \det A & \\ 0 & & \ddots \; \det A \end{bmatrix}$$

cofactors of last row

② · ③ => diagonals are just cofactor formula restated

non-diagonals are blank because rows/cols stop being independent when we misalign cofactors anyway $AC^T = \det A\, I$:) ✓

$Ax = b$

$x = A^{-1} b = \underbrace{\frac{1}{\det A}}_{\text{what are easy ways to calc?}} C^T b$ ↗ $C_{11}b_1 + C_{21}b_2 + \ldots$

CRAMER'S RULE

$x_1 = \dfrac{\det B_1}{\det A}$ $\qquad B_1 = \begin{bmatrix} | & n-1 \\ b & \text{of} \\ | & A \end{bmatrix}$

$x_n = \dfrac{\det B_n}{\det A}$ $\qquad B_n$ = matrix $\therefore$ A w/ column n replaced by b!

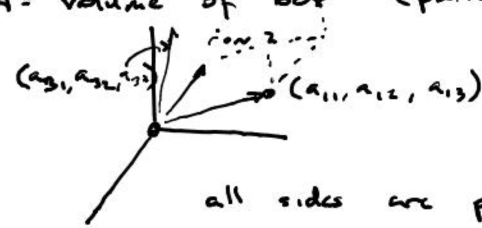
IS THIS EASIER THAN ELIMINATION? MAYBE NOT! :>.

BUT IT'S ALGEBRAIC RATHER THAN COMPUTATIONAL

DETERMINANT AS VOLUME!

det A = volume of box (parralelipiped)

$\mathbb{R}^3$



all sides are parallelograms

det A is negative? change in chirality

$A = I \Rightarrow$ unit cube

can prove by showing box follows same rules as det orthogonal matrix $A = Q$

$\det A = \pm 1$ (unit cube rotated)

$Q^T Q = I \qquad |Q^T Q| = |I|$

$$|Q^T||Q| = 1$$

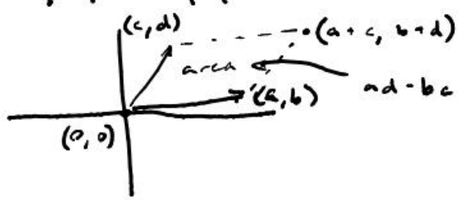
$$|Q|^2 = 1 \quad :)$$

cube case solved! let's stretch edges for orthogonal

double first row of matrix → by rule 3a det doubles (and so does volume)

$|\det A|$ = volume of box ← 1 ✓, 2 ✓, 3a ✓, 3b ?

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$



▱ Area = $\det A = ad - bc$

△ $= \frac{1}{2}\det A = \frac{ad-bc}{2}$

triangle doesn't start at origin

$$\text{area} = \frac{1}{2}\left[(x_2 - x_1)(y_3 - y_1) - (x_2 - x_1)(y_2 - y_1)\right]$$

LECTURE 21 EIGENVALUES & EIGENVECTORS

$$\det[A-\lambda I]=0$$

$$\text{TRACE}: \lambda_1+\lambda_2+\ldots+\lambda_n=0$$

EIGENVALUES & EIGENVECTORS TAKES UP MOST OF THE REST OF THE COURSE. WHAT ARE THEY?

EIGENVECTORS FIRST. CONSIDER A MATRIX $\underline{A}$. WHAT DOES IT DO? IT'S A FUNCTION THAT ACTS ON VECTORS!

MOST VECTORS ACTED ON BY A MATRIX CHANGE DIRECTIONS. BUT $\boxed{Ax=\lambda x}$ FOR SOME SPECIAL X. λ IS A LINEAR MULTIPLICATION OF x; WHAT ABOUT $\lambda=0$? IF A IS SINGULAR, $\lambda=0$ IS A EIGENVALUE,

LET'S TAKE A PROJECTION MATRIX

WHAT ARE X AND λs FOR A PROJECTION MATRIX? ANY X IN PROJECTION PLANE: x; $\lambda=1$

$Px=0x$ w/ $\lambda=0$ too – orthogonal!

PERMUTATION MATRIX

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\lambda = 1$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $x_2 = x_1$

$\lambda = -1$, $x_2 = -x_1$

I ASSUME NORMAL EIGENVALUES ARE BETTER?

$n \times n$ matrix will have n eigenvalues. the sum of the eigenvalues is the sum of the diagonals - the trace $a_{11} + a_{22} + \dots + a_{nn} = 0$ in this case! so knowing $n-1$ eigenvalues gives you the last for free (but λs can repeat)

BACK TO $Ax = \lambda x$ ← HOW TO SOLVE

we have two unknowns but rewrite...

$(A - \lambda I)x = 0$

if x exists, $A - \lambda I$ must be singular; $\det(A - \lambda I) = 0$ ①

① "THE CHARACTERISTIC EQUATION" LETS YOU FIND λs FIRST. THEN FIND x BY ELIMINATION w/ FREE VARIABLES ETC

EXAMPLE $A: \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ - note symmetric matrix gives you real eigenvalues & orthogonal xs

$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = \lambda^2 - \overset{①}{6}\lambda + \overset{②}{8}$

① TRACE(A) ② DET(A) $\therefore \lambda_1 = 2$ $\lambda_2 = 4$

$\lambda_2 = 4$ first

$$A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}; \quad x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_1 = 2$

$$A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}; \quad x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

COMPARE TO PERMUTATION EXAMPLE – OUR A WAS $P + 3I$. SO WE JUST ADDED 3 TO EACH EIGENVALUE!

$$Ax = \lambda x$$

$$(A + 3I)x = (\lambda + 3I)x \quad (\text{if } x)$$

NOT SO GREAT/EASY

if $Ax = \lambda x$, B has eigenvalues α_1 & α_2 what happens to $A+B$? DOES $(A+B)x = (\lambda + \alpha)x$? NO. WHY? because B & A usually have different eigenvectors so adding them makes no sense. so linear combination doesn't work unless x consistent. which it won't be

EXAMPLE $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ trace $= 0 + 0 = \lambda_1 + \lambda_2$

90° rotation det $= 1 = \lambda_1 \cdot \lambda_2$

$\hookrightarrow$ obv complex

$$\det(Q - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0, \quad \lambda_1 = i; \ \lambda_2 = -i$$

SYMMETRIC MATRICES GIVE REAL EIGENVALUES

ANTI-SYMMETRIC " IMAGINARY "

ANOTHER BAD EXAMPLE

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \Rightarrow \det(A - \lambda I) = (3-\lambda)(3-\lambda); \quad \lambda_1 = 3, \lambda_2 = 3$$

if matrix is triangular the eigenvalues are on diagonal

① BUT WHAT ARE EIGENVECTORS?

$$(A - \lambda I)x = 0 \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = 0 \qquad x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot 3$$

$x_2 = ??$ $\leftarrow$ doesn't exist

repeated eigenvalues can mean dependent eigenvectors, uh-oh

LECTURE 22 DIAGONALISATION; POWERS OF A

LET'S USE WHAT WE LEARNED IN L21

· DIAGONALISING A MATRIX: $S^{-1}AS = \Lambda$ ①

eigenvector matrix, needs n independent eigenvectors of A

suppose we satisfy ①. put them in columns of S

$$AS = A\begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_1 x_1 & \cdots & \lambda_n x_n \\ | & & | \end{bmatrix} = \begin{bmatrix} x_1 & \cdots & x_n \\ | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & - & 0 \\ | & \ddots & | \\ 0 & - & \lambda_n \end{bmatrix}$$

why you need independent eigenvectors

diagonal eigenvalue matrix Λ

$AS = S\Lambda \Rightarrow S^{-1}AS = \Lambda$

$\Rightarrow A = S\Lambda S^{-1}$

consider A^2

substitute

if $Ax = \lambda x$, $A^2x = \lambda Ax = \lambda^2 x$. λs of $A^2 = \lambda^2$ of A

how can we see from $A = S\Lambda S^{-1}$

$$A^2 = S\Lambda \underbrace{S^{-1}S}_{I}\Lambda S^{-1}$$

$$= S\Lambda^2 S^{-1}$$ ← multiplication & addition work fine if S is sane :)

$A^k = S\Lambda^k S^{-1}$ ← eigenvalues & vectors let us understand the powers of a matrix; they're preserved. we previously had no mechanism to look at this

THEOREM: $A^k \to 0$ as $k \to \infty$ if all $|\lambda_i| < 1$

assuming you have independent eigenvectors

A MATRIX IS SURE TO HAVE n INDY EIGENVECTORS AND BE DIAGONALISABLE IF NO EIGENVALUES REPEAT

eig(rand(10,10)) => 10 distinct λs

REPEATED EIGENVALUES → may or may not have n indy eigenvectors

$n = 10$ $\quad I = \lambda_1 \ldots \lambda_{10} = 1 \ldots$ lots of eigenvectors!

but suppose $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ $\quad \lambda_1 = 2,\ \lambda_2 = 2,\quad x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \dim(N(A)) = 1$, 1 eigenvector

algebraic multiplicity is 2, geometric is 1. not diagonalisable.

BACK TO SAFETY: START w/ given vector u_0

$u_{k+1} = Au_k \qquad u_1 = Au_0,\ u_2 = A^2u_0,\ u_k = A^ku_0$

to really solve $u_k = A^k u_0$, write u_0 as combination of eigenvectors. $u_0 = c_1 x_1 + c_2 x_2 + \dots + c_n x_n = Sc$

← clever

$$Au_0 = \lambda_1 c_1 x_1 + \dots + \lambda_n c_n x_n$$

$$A^{100} u_0 = \lambda_1^{100} c_1 x_1 + \dots + \lambda_n^{100} c_n x_n$$

$$= \Lambda^{100} S c$$

FIBONACCI EXAMPLE

$0, 1, 1, 2, 3, 5, 8 \dots \qquad F_1 = 0 \quad F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n$

$F_n = ??$

oh my god!

$$F_{k+2} = F_{k+1} + F_k$$
$$F_{k+1} = F_{k+1}$$

$$u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} \qquad u_{k+1} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}}_{u_k}$$

eigenvalues & vectors of A

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = (1-\lambda)(-\lambda) - 1 = \overbrace{\lambda^2 - \lambda - 1}^{\text{recursion appearing here}} = 0$$

$$\lambda = \frac{1 \pm \sqrt{1+4}}{2} \Rightarrow \lambda_1 = \tfrac{1}{2}(1+\sqrt{5}) \sim 1.618 \quad ①$$

$$\lambda_2 = \tfrac{1}{2}(1-\sqrt{5}) \sim -0.618$$

① controlling growth $\Rightarrow F_{100} \approx c_1 \lambda_1^{100}$ (neglect λ_2 term)

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \qquad x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} \qquad u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

when things are evolving by a first order system, find eigen A, u_0 as combos of xs

EXTRAS – LET'S FINISH FIBONACCI

$x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$ $u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$u_0 = c_1 \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} \Rightarrow u_n = \frac{1}{\sqrt{5}} \lambda_1^n \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{5}} \lambda_2^n \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$

$u_n = (\lambda_1^n - \lambda_2^n)/\sqrt{5}$

EXTRAS – FULL PSEUDO-INVERSE

$(A^T A)^{-1} x = A^T$ is a way to make non-square matrices w/ independent columns sort-of invertable; you've used this in 4fd plenty. what about pseudo-inverting singular square matrices?

the full pseudo-inverse A^+:

- if vector is in colspace, preserved by A^+
- if vector is collapsed, A^+ sets to 0

then uses singular value decomposition w/

$A = U \Sigma V^T$; U & V are orthogonal "factors" of A

Σ is a diagonal matrix of singular values

JORDAN NORMAL FORM – "generalised eigenvectors"

EXTRAS: PROOF THAT DISTINCT EIGENVALUES LEAD TO INDEPENDENT EIGENVECTORS ALWAYS

SUPPOSE x_1 and x_2 w/ distinct eigenvalues are dependent. $x_2 = c x_1 \Rightarrow A x_2 = \lambda_2 x_2 \Rightarrow A c x_1 = \lambda_2 c x_1$

$$c\lambda_1 x_1 = c\lambda_2 x_1$$

$$\lambda_1 = \lambda_2$$

not distinct!

$\therefore$ SHOWN BY CONTRADICTION

LECTURE 23 DIFFERENTIAL EQUATIONS $\frac{du}{dt} = Au$

e^{At} OF A MATRIX

EXAMPLE $du_1/dt = -u_1 + 2u_2$ $\quad du_2 = u_1 - 2u_2$ $\quad u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \qquad \lambda_1 = 0 \qquad \lambda_2 = -3 \text{ (from trace or } \lambda^2 + 3\lambda = 0\text{)}$$

① need eigenvectors but we know solution will have two parts - an e^{0t} & e^{-3t}. we therefore have a grip on long-term behavior

① $x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\qquad A x_1 = \lambda_1 x_1$

$$A x_2 = \lambda_2 x_2$$

②

SOLUTION: $u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$

CHECK $du/dt = Au$; plug in $e^{\lambda t} x_1$ $\lambda_1 e^{\lambda_1 t} x_1 = A e^{\lambda t} x_1$

③ PURE EXPONENTIALS ARE GENERAL SOLUTIONS

SOLVE c_1 & c_2 AFTER PLUGGING IN λ, x

$= c_1 \cdot 1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

NEED u_0 $[1 \ 0] \Rightarrow @ \ t=0$ $\quad c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$c_1 = 1/3 \quad c_2 = 1/3$

$u(t) = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3} e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

steady state w/ $u \to \infty = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

we don't always have a steady state. when do we have stability?

$u(t) \to 0$ when real part of eigenvalues negative

when do we have steady state?

$u(t) \to c$ when $\lambda_1 = 0$ & real part of other eigenvalues is negative

$u(t) \to \infty$ when any real part of eigenvalue is positive

FOR 2x2 MATRIX WE CAN TELL WHETHER REAL PART OF EIGENVALUES IS NEGATIVE IF

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{array}{ll} \text{trace: negative} & (\lambda_1 + \lambda_2) \\ \text{det: positive} & (\lambda_1 \cdot \lambda_2) \end{array}$$

COMPLEX λs WILL BE CONJUGATES

BACK TO EXAMPLE EQUATION

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{c} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{= u^0}$$

A couples the system u_1 u_2

λ & x uncouple — diagonalise them

CAN WE DO THIS IN TERMS OF S AND λ?

$du/dt = Au$ SET $u = Sv$ ← v is in eigenvector coordinates

$S\, dv/dt = ASv$

$dv/dt = S^{-1}ASv = \Lambda v$!

$dv/dt = \lambda_1 v_1 \ldots \lambda_n v_n$

$v(t) = e^{\Lambda t} v(0)$

$u(t) = \underbrace{S e^{\Lambda t} S^{-1}}_{①} u(0)$

① $e^{At} = S e^{\Lambda t} S^{-1}$

WHAT DOES e^{At} EVEN MEAN?

WHAT IS THE EXPONENTIAL OF A MATRIX?

USE POWER SERIES!

① $e^{At} = I + At + \frac{(At)^2}{2} + \dots + \frac{(At)^n}{n!}$ ← TAYLOR SERIES e^x

② $(I - At)^{-1} = I + At + (At)^2 \dots + (At)^n$ 〃 $1/(1-x)$

REMEMBER TAYLOR FOR e^{At} ALWAYS CONVERGES. FOR ② NEED $|\lambda(A(t))| < 1$

HOW DO WE CONNECT $Se^{\Lambda t}S^{-1}$ to e^{At}?

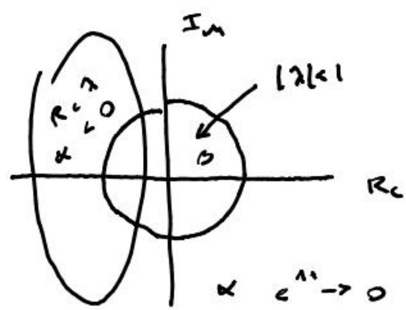
$$e^{At} = I + S\Lambda S^{-1} + \frac{S\Lambda^2 S^{-1}}{2} + \dots + \frac{S\Lambda^n S^{-1}}{n!}$$

make use of what we learned about powers

$$e^{At} = S\left(1 + \Lambda t + \Lambda^2 t^2/2 + \dots + \Lambda^n t^n/n!\right)S^{-1} = Se^{\Lambda t}S^{-1}$$ for independent eigenvectors

what is $e^{\Lambda t}$? $\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$$



α $e^{\Lambda t} \to 0$

β $A^k \to 0$

LAST EXAMPLE

$$y'' - by' - Ky = 0$$

change 1 second-order eq into system

$$u = \begin{bmatrix} y' \\ y \end{bmatrix} \quad u' = \begin{bmatrix} y'' \\ y' \end{bmatrix} = \begin{bmatrix} -b & -K \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix}$$

coefficients

5^{th} order to 5×5 1^{st}-order $\begin{bmatrix} - & - & - & - & - \\ 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \end{bmatrix}$ ← COMPANION MATRIX

EXTRAS: CALC λ, x FOR

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \Rightarrow \begin{vmatrix} -1-\lambda & 2 \\ 1 & -2-\lambda \end{vmatrix} \Rightarrow (-1-\lambda)(-2-\lambda) - 2 = 0$$

$$\lambda^2 - 3\lambda = 0$$

$$\lambda = 0, -3$$

SOLVE $x \Rightarrow \lambda_1 = 0$ so

$$\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \text{ by elimination; rref} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$$x_1 = [2, 1]$$

$$\lambda_2 = -3 \Rightarrow \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow x_2 = [1 \; -1]$$

LECTURE 24 APPLICATIONS OF EIGENVALUES

MARKOV MATRICES

FOURIER SERIES

MARKOV MATRIX

$$A = \begin{bmatrix} .1 & .01 & .3 \\ .2 & .99 & .3 \\ .7 & 0 & .4 \end{bmatrix}$$

① ALL ENTRIES ≥ 0

② COLUMNS SUM TO 1

BOTH PROPERTIES RETAINED AT ANY A^k

IN POWERS, $\lambda = 1$ LEADS TO STEADY STATE

ASSERTION: MARKOV MATRICES HAVE AN $\lambda = 1$ ③

ALL OTHER $|\lambda_i| < 1$

$$u_k = A^k u_0 = \underbrace{c_1 \overset{\text{if } 1}{\lambda^k} x_1 + \ldots + c_n \overset{\text{if } < 1}{\lambda_n^k} x_n}_{\text{approaches } c_1 x_1}$$

← assume independent eigenv.

WHY DOES ② LEAD TO ③?

$$A - I = \begin{array}{ccc} 0 & 0 & 0 \\ \left[\begin{matrix} -0.9 & 0.01 & 0.3 \\ 0.2 & -0.01 & 0.3 \\ 0.7 & 0 & -0.6 \end{matrix}\right] \end{array}$$

show this is singular, then $1 = \lambda_1$

SHOW ROWS ARE DEPENDENT BY ADDING THEM :)

$R_1 + R_2 + R_3 = 0$

$(1, 1, 1)$ is in $N(A^T)$

$\therefore \lambda_1 = 1$

x_1 is in $N(A)$!

find x_1

$$\begin{bmatrix} -0.9 & 0.01 & 0.3 \\ 0.2 & -0.01 & 0.3 \\ 0.7 & 0 & -0.6 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

eigenvalues of A & A^T are the same (coordinate transform!). $\det|A - \lambda I| = 0$
$= \det|A^T - \lambda I|$ ✓

$$\left.\begin{array}{l} x_{11} = 0.6 \\ x_{12} = 33 \\ x_{13} = 0.7 \end{array}\right\} \text{all positive.}$$

calculating unpleasant

APPLICATION OF MARKOV MATRICES

$u_{k+1} = A u_k$ w/ A as MARKOV

$$\begin{bmatrix} u_{CAL} \\ u_{MASS} \end{bmatrix}_{t=k+1} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} u_{CAL} \\ u_{MASS} \end{bmatrix}_{t=k}$$

90% of californians move

80% of mass. ppl move

$$\begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \qquad \lambda_1 = 1 \quad \lambda_2 = 0.7 \qquad \begin{bmatrix} u_{CAL} \\ u_{MASS} \end{bmatrix}_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$$

$$\downarrow$$

$$\lambda_1 = 1 ; \begin{bmatrix} -0.1 & 0.2 \\ 0.1 & -0.2 \end{bmatrix} \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{x_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} u_{CAL} \\ u_{MASS} \end{bmatrix}_1 = \begin{bmatrix} 200 \\ 800 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} u \end{bmatrix}_2 = \begin{bmatrix} 340 \\ 660 \end{bmatrix}$$

$$\downarrow$$

x_1 gives steady state :)

$$x_2 \Rightarrow \begin{bmatrix} 0.2 & 0.2 \\ 0.1 & 0.1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} u \end{bmatrix}_{100} = ?$$

$$A^k = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 0.7^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

solve c_1 & c_2 w/ $u_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$

$$A^{100} = \frac{1000}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{2000}{3} 0.7^{100} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad c_1 = 1000/3 ; \; c_2 = 2000/3$$

MARKOV MODELS MOVEMENT BETWEEN NODES w/ NO LOSS.

FOURIER SERIES & PROJECTIONS

projections w/ orthonormal basis

$$q_1 \cdots q_n$$

①

$v = x_1 q_1 + x_2 q_2 + \dots + x_n q_n$

isolate x_1 by taking inner product w/ q_1. all

$q_{!1}$ terms go to zero b/c orthonormal

$\therefore q_1^T v = x_1 \underbrace{q_1^T q_1}_{\to 1} + 0 + \dots + 0$

① $\begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = v \qquad Qx = v; \quad x = Q^{-1} v$ ②

② B/C ORTHONORMAL, $Q^{-1} = Q^T$

$\therefore x_1 = q_1^T v$. matches :)

FOURIER SERIES ARE BUILT ON ORTHONORMAL BASES

$f(x) = a_0 + a_1 \cos(x) + b_1 \sin(x) + a_2 \cos(2x) + b_2 \sin(2x)$

orthogonal functions in infinite dimensional space

bases: $1, \cos x, \sin x, \cos 2x, \sin 2x$

WHAT DOES IT MEAN FOR FUNCTIONS TO BE ORTHOGONAL? INNER PRODUCT = 0

VECTORS: $v^T w = v_1 w_1 + \dots + v_n w_n$

FUNCTIONS: $f(x)^T g(x) \Rightarrow$ multiply $f(x) \cdot g(x)$ at every x

integrate

$$f^T g = \int_0^{2\pi^{①}} f(x) g(x)\, dx = 0$$

① PERIODIC AROUND UNIT CIRCLE FROM SIN & COS

$$\int_0^{2\pi} \sin x \cos x \, dx = \frac{1}{2} \sin x^2 \Big|_{0 \to 2\pi} = 0$$

how do we get fourier coefficients?

for a_1, inner product of everything w/ cos x

$$\int_0^{2\pi} f(x) \cos x \, dx \rightarrow \text{all non } \cos x \text{ terms vanish}$$

$$= \frac{1}{\pi} \int_0^{2\pi} (\cos x)^2$$

PROBLEM SETS FROM UNIT 2

1) FOR EVERY SYSTEM OF m EQUATIONS WITH NO SOLUTION, THERE ARE NUMBERS $y_1, \dots, y_m$ THAT MULTIPLY THE EQUATIONS SO THEY ADD UP TO $0 = 1$.

EXACTLY ONE OF THESE PROBLEMS HAS

A SOLUTION:

$Ax=b$ OR $A^Ty=0$ with $y^Tb=1$

IF b IS NOT IN THE COLUMN SPACE OF A IT IS NOT ORTHOGONAL TO THE NULLSPACE OF A^T.

MULTIPLY $x_1-x_2=1$, $x_2-x_3=1$ and $x_1-x_3=1$ by y, b CHOSEN TO FORCE $0=1$

$$A=\begin{bmatrix}1&-1&0\\0&1&-1\\1&0&-1\end{bmatrix} \qquad A^T=\begin{bmatrix}1&0&1\\-1&1&0\\0&-1&-1\end{bmatrix}$$

we know $Ax=b$ is not going to work, dependent rows in A. so $A^Ty=0$ with $y^Tb=1$

y is in left nullspace

$A^Ty=0$; $y^Tb=1$

$$A^T=\begin{bmatrix}1&0&1\\-1&1&0\\0&-1&-1\end{bmatrix} \Rightarrow \begin{bmatrix}1&0&1\\0&1&1\\0&-1&-1\end{bmatrix} \Rightarrow \begin{bmatrix}1&0&1\\0&1&1\\0&0&0\end{bmatrix} \Rightarrow y=\begin{bmatrix}-1\\-1\\1\end{bmatrix}$$

$(x_1-x_3)-(x_1-x_2)-(x_2-x_3) \Rightarrow x_1-x_1-x_3+x_3+x_2-x_2=0$

$y^Tb=1$ for $b=\begin{bmatrix}0\\0\\1\end{bmatrix}$ ✓

2. SUPPOSE WE HAVE FOUR NONZERO VECTORS r, n, c and l in $\mathbb{R}^2$

a) what are conditions for these to be bases of the four fundamental subspaces of a 2×2 matrix

- orthogonal pairings
- in those pairings, one vector describes nullspace, other describes dim $\mathbb{R}^2$ - nullspace
- matrix must have rank 1

NB: although we have letters corresponding to row, column, nullspace we're not obligated here

b) any vector by a transpose of another vector will construct a valid A here although the terms might get weird :)

3. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ project $b = (1,2,3,4)$ onto the colspace of A. what shape is the projection matrix P; what is P?

$P = A(A^TA)^{-1}A^T \rightarrow$ THIS WILL BE 4×4 with

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$$

4. IF $P^2 = P$, show $(I-P)^2 = I - P$. ①

FOR A & P FROM PREVIOUS QUESTION, P PROJECTS ONTO C(A) AND I-P PROJECTS ONTO ____ ?
②

① $(I-P)^2 \Rightarrow I^2 - 2P + \underbrace{P^2}_{=P} \Rightarrow \underbrace{I^2}_{=I} - P \Rightarrow I - P$

② LEFT NULLSPACE OF A   I-P IS 𝓡

5. WRITE DOWN 3 EQUATIONS FOR $b = C + Dt$

TO GO THROUGH $b = 7$ @ $t = -1$, $b = 7$ @ $t = 1$, $b = 21$ @ $t = 2$

FIND THE LEAST SQUARES SOLUTION AND PLOT

CLOSEST LINE.

$$\underbrace{\begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}}_{b} = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \cdot \begin{bmatrix} C \\ D \end{bmatrix}$$

① $\left[\begin{array}{cc|c} 3 & 2 & 35 \\ 2 & 6 & 42 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 3 & 2 & 35 \\ 1 & 3 & 21 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & 3 & 21 \\ 3 & 2 & 35 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 3 & 21 \\ 0 & -7 & -28 \end{array}\right]$

$D = 4 \quad C = 21 - 12 = 9$

$$A^T b = (A^T A) \cdot \hat{x}$$

$$= \left(\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \right) = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix} \Rightarrow \begin{bmatrix} 35 \\ 42 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} \Rightarrow Ax = b \text{ solve } x$$

①

6 FIND $p = A\hat{x}$ of previous problem. show $e = (2, -6, 4)$

why is $P(e) = 0$

$$p = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 9 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 17 \end{bmatrix}$$

$$\left.\begin{array}{l} 7 - 2 \\ 7 - -6 \\ 21 - 4 \end{array}\right\} e = \begin{bmatrix} 2 \\ -6 \\ 4 \end{bmatrix}$$

$Pe = 0$ because e is orthogonal to proj plane

7. compute $\hat{x}$ for $b = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

$\hat{x} : 0$ because orthogonal

8. $\hat{x} = x,\ e = 0$ because we're in the projection plane

9. WHICH OF THE FOUR SUBSPACES CONTAINS e? ①

WHAT ABOUT p? ②

WHAT ABOUT $\hat{x}$? ③

WHAT IS THE NULLSPACE OF A? ④

① $N(A^T)$

② $C(A)$

③ $R(A)$

④ $\{0\}$

10. $b = \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ $A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{bmatrix}$

$A^T b = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$ $A^T A = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$

$\left[\begin{array}{cc|c} 5 & 0 & 5 \\ 0 & 10 & -10 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right] \Rightarrow C = 1 \quad D = -1$

$y = 1 - t$

11. ORTHONORMAL VECTORS ARE LINEARLY INDEPENDENT.

SHOW $Qx=0$ IMPLIES $x=0$. SINCE Q MAY BE RECTANGULAR YOU CAN USE Q^T BUT NOT Q^{-1}

$$\underbrace{Q^TQ}\,x=0$$

$$Ix=0 \quad \therefore \; x=0$$

12. GIVEN THE VECTORS a, b, c LISTED BELOW, USE GRAM-SCHMIDT TO FIND ORTHOGONAL VECTORS A B C THAT SPAN THE SAME SPACE

$$a=(1,-1,0,0) \quad b=(0,1,-1,0) \quad c=(0,0,1,-1)$$

$$A=a$$

$$B=b-A^Tb\,(A^TA)^{-1}A = ①$$

$$C=c-A^Tc\,(A^TA)^{-1}A-B^Tc\,(B^TB)^{-1}B = ②$$

$$① \quad B=\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}-\overset{[-1]}{A^Tb}\,\overset{[1/2]}{(A^TA)^{-1}}A=\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}+\begin{bmatrix}1/2\\-1/2\\0\\0\end{bmatrix}=\begin{bmatrix}1/2\\1/2\\-1\\0\end{bmatrix}$$

$$② \quad c=\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}-\cancel{\overset{0}{A^Tc}\cdot 1/2}\cdot A-\underset{[-1]}{B^Tc}\,\underset{③}{(B^TB)^{-1}}B$$

$$c=\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}+\frac{2}{3}\begin{bmatrix}1/2\\1/2\\-1\\0\end{bmatrix}=\begin{bmatrix}1/3\\1/3\\1/3\\-1\end{bmatrix}$$

$$③ = 1/2\cdot 1/2+1/2\cdot 1/2+-1\cdot -1+0=(3/2)^{-1}$$

SHOW THAT $\{A,B,C\}$ and $\{a,b,c\}$ are bases for the SPACE OF VECTORS PERPENDICULAR TO $d=(1,1,1,1)$

SHOW $A \cdot d = 0$, $B \cdot d = 0$, $C \cdot d = 0$

$-1+1=0 \qquad -1+1=0 \qquad -1+1=0$

$\therefore \{A, B, C\}$ all orthogonal to d, orthonormal basis of vector space in $\mathbb{R}^4$. $\{a, b, c\}$ inhabit this space & are indep. so they are also a basis

13. IF ENTRIES IN EVERY ROW SUM TO 0, COLUMNS NOT INDEPENDENT. SINGULAR MATRICES HAVE DET 0

IF ALL SUM TO 1, SUBTRACT I TO GET THEM SUM TO ZERO AGAIN SO $\det(A-I)=0$

14. USE ROW OPERATIONS & PROPERTIES OF THE DETERMINANT TO CALC

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \Rightarrow \text{product of pivots so}$$

$$\begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

$$(b-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & c-a & c^2-a^2 \end{vmatrix} \Leftarrow$$

$$(b-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & c-a & c^2-a^2 \end{vmatrix} \Rightarrow \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & ① \end{vmatrix}$$

$$① = (c^2 - a^2) - (c-a)(b+a)$$
$$= c^2 - \cancel{a^2} - cb + ca - ab + \cancel{a^2}$$
$$= c^2 - cb - ca - ab = (c-a)(c-b)$$

pull out to get $(b-a)(c-a)(c-b) \det I$ ✓

15. $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $|A| = -1$, triple row switch & sum by inspection

16. THE SYMMETRIC PASCAL MATRICES HAVE DET 1. IF WE SUBTRACT 1 FROM THE n, n ENTRY, WHY DOES THE DETERMINAT BECOME 0?

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix} = 1 \quad \text{(KNOWN)}$$

$$A' = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 19 \end{vmatrix} = 0 \quad \text{(EXPLAIN)}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix} \quad (n-1)\times(n-1) \text{ COFACTOR DET} = 1$$

$$\det A' = - \overset{1}{20}\, \overset{1}{c_{nn}(A)} \underbrace{- 10\, c_{n-1\,n}(A) + 4\, c_{n-2\,n}(A) - c_{n-3\,n}(A)}_{\text{constant } B}$$

$1 = 20 + 1 - B$

$B = 20$

$\det A' = 19 + 1 - B = 0$

17. SUPPOSE $A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}$. FIND ITS COFACTOR MATRIX AND MULTIPLY AC^T TO FIND $\det(A)$

$$C = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix} \quad C^T = \begin{bmatrix} 6 & 3 & -6 \\ -3 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$$

$$AC^T = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 6 & 3 & -6 \\ -3 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow \det(A) = 3$$

if you change a_{13} from $4 \to 100$, you're still multiplying it by its cofactor 0 so the sum won't change

18. spherical coordinates ρ, ϕ, θ satisfy

$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$

find its 3x3 matrix of partial derivatives

$$A = \begin{bmatrix} \partial x/\partial \rho & \partial x/\partial \phi & \partial x/\partial \theta \\ \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \partial y/\partial \rho & \partial y/\partial \phi & \partial y/\partial \theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \partial z/\partial \rho & \partial z/\partial \phi & \partial z/\partial \theta \\ \cos\phi & -\rho\sin\phi & 0 \end{bmatrix}$$

now calculate $\det(A)$

$$\underset{①}{\cos\phi \, C_{31}} + \underset{②}{\rho \sin\phi \, C_{32}} + 0$$

① $\rho\cos\phi\cos\theta \cdot \rho\sin\phi\cos\theta - \rho\sin\phi\sin\theta \cdot \rho\cos\phi\sin\theta$

$\rho^2\cos^2\theta\cos\phi\sin\phi - \rho^2\sin^2\theta\sin\phi\cos\phi$

$\rho^2\cos\phi\sin\phi(\cos^2\theta + \sin^2\theta)$

② $\sin\phi\cos\theta \cdot \rho\sin\phi\cos\theta - -\rho\sin\phi\sin\theta \cdot \sin\phi\sin\theta$

$\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta$

$\rho\sin^2\phi(\cos^2\theta + \sin^2\theta)$

① $\cdot\cos\phi$ + ② $\rho\sin\phi = \det(A)$

$-\rho^2\cos^2\phi\sin\phi(\overset{1}{\cancel{\cos^2\theta - \sin^2\theta}}) + \rho^2\sin^3\phi(\overset{1}{\cancel{\cos^2\theta + \sin^2\theta}})$

$\rho^2\sin\phi(\overset{1}{\cancel{\sin^2\phi + \cos^2\phi}})$

$\rho^2\sin\phi$ ✓

19. A 3×3 MATRIX B IS KNOWN TO HAVE $\lambda = 0, 1, 2$

CAN WE FIND: a) RANK B = 2

b) DET $B^T B$ = 0

c) EIGENVALUES $B^T B$ ×

d) " $(B^2 + I)^{-1}$

$\frac{1}{0+1}, \frac{1}{1^2+1}, \frac{1}{2^2+1} \Rightarrow 1, \frac{1}{2}, \frac{1}{5}$

20. $A=\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ $B=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$ $C=\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

FIND λs

$\lambda_A = (1,4,6)$ by inspection

$\lambda_B = |B-\lambda I|$

$$\begin{matrix} -\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 3 & 0 & -\lambda \end{matrix}$$

$$-\lambda(2-\lambda)-\lambda + 1\cdot(0-(2-\lambda)3)$$

$$\lambda^2(2-\lambda) - 3(2-\lambda)$$

$$(2-\lambda)(\lambda^2-3) = 0$$

$$\lambda = 2,\ \lambda = \sqrt{3}$$

$\lambda_C =$ 0, 0 $\leftarrow$ BY RANK

$\lambda_3 = 6$ $\leftarrow$ BY TRACE

21. DESCRIBE ALL MATRICES S THAT DIAGONALISE $A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}$, THEN FIND S FOR A^{-1}

①

$A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}$ $|A - \lambda I| = \begin{vmatrix} 4-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix}$

$D = (4-\lambda)(2-\lambda)$

$\lambda_1 = 4 \ , \ \lambda_2 = 2$

$x_1 \Rightarrow \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ 1 \end{bmatrix} = 0 \Rightarrow x_{11} = 2x_{12} \Rightarrow [2, 1]$

$x_2 \Rightarrow \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ 1 \end{bmatrix} = 0 \Rightarrow 2x_{21} = 0x_{22} \Rightarrow [0, 1]$

$S = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ for A and A^{-1} $(A^{-1} = S\Lambda^{-1}S^{-1})$

22. FIND Λ AND S TO DIAGONALISE A

$A = \begin{bmatrix} 0.6 & 0.9 \\ 0.4 & 0.1 \end{bmatrix}$ $\lambda_1 = 1 , \ \lambda_2 = -0.3$

$x_1 \Rightarrow \begin{bmatrix} -0.4 & 0.9 \\ 0.4 & -0.9 \end{bmatrix} \cdot \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} -4x_{11} - 9x_{12} = 0 \Rightarrow x_{11} = \frac{9}{4} x_{12}$

$x_1 = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$

$x_2 \Rightarrow \begin{bmatrix} 0.9 & 0.9 \\ 0.4 & 0.4 \end{bmatrix} \cdot \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_{21} + x_{22} = 0 \Rightarrow x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$S = \begin{bmatrix} 9 & 1 \\ 4 & -1 \end{bmatrix}$ $\lambda = \begin{bmatrix} 1 & 0 \\ 0 & -0.3 \end{bmatrix}$, $k \Rightarrow \infty$, $\Lambda^k \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

23. $A^T = -A$ $\quad du/dt = \begin{bmatrix} 0 & c & b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} u$

FIND THE DERIVATIVE OF $\|u(t)\|^2 = u_1^2 + u_2^2 + u_3^2$

$$d\|u(t)\|^2/dt = 2u_1u_1' + 2u_2u_2' + 2u_3u_3'$$

$$= 2\left[u_1(cu_2+bu_3) + u_2(au_3-cu_1) + u_3(bu_1-au_2)\right]$$

$$= 2\left[cu_1u_2 - cu_1u_2 + au_2u_3 - au_2u_3 + bu_3u_1 - bu_3u_1\right]$$

$$= 0$$

OR: $\|u(t)\|^2 = u^Tu$

$$d/dt\,(u^Tu) = (u')^Tu + u^Tu'$$

$$= (Au)^Tu + u^T(Au)$$

$$= u^TA^Tu + u^TAu$$

$$= -u^TAu + u^TAu \Rightarrow 0$$

$u(t)$ constant

24. WRITE $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ as $S\Lambda S^{-1}$. MULTIPLY $Se^{\Lambda t}S^{-1}$ to find e^{At}. CHECK YOUR WORK BY EVALUATING e^{At} AND THE DERIVATIVE of e^{At} AT $t=0$

λ_s: $\quad |A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} \Rightarrow (1-\lambda)(3-\lambda)=0$

$\lambda = 1, 3$

$x_1 \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$x_2 \Rightarrow \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = -2x_{21} = -1x_{22} \Rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$S = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \quad S^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$

$$Se^{\Lambda t}S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} e^{t} & 0 \\ 0 & e^{3t} \end{bmatrix}\frac{1}{2}\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\hookrightarrow \begin{bmatrix} e^{t} & \frac{1}{2}(e^{3t}-e^{t}) \\ 0 & e^{3t} \end{bmatrix}$$

$$e^{At} @ t=0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\frac{de^{At}}{dt} = \begin{bmatrix} e^{t} & \frac{1}{2}(3e^{3t}-e^{t}) \\ 0 & 3e^{3t} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = A \quad ✓$$

LECTURE 25: SYMMETRIC MATRICES

EIGENVALUES / EIGENVECTORS

POSITIVE DEFINITE MATRICES

$A = A^T$ ① THE EIGENVALUES ARE REAL

② THE EIGENVECTORS ARE ORTHONORMAL

CAN BE CHOSEN TO BE

USUAL CASE $A = S \Lambda S^{-1}$

SYMMETRIC CASE $A = Q \Lambda Q^{-1}$ BUT $Q^{-1} = Q^T$

$$A = Q \Lambda Q^T = A^T$$

SPECTRAL THEOREM WHERE λ IS A "SPECTRUM"

or PRINCIPAL AXIS THEOREM

① WHY IS THIS TRUE? TAKE $Ax = \lambda x \Rightarrow \bar{A}\bar{x} = \bar{\lambda}\bar{x}$ always can take complex conj.

BUT WE'RE USING REAL MATRICES SO

$$A\bar{x} = \bar{\lambda}\bar{x} \Rightarrow \bar{x}^T A^T = \bar{x}^T \bar{\lambda} \Rightarrow \bar{x}^T A x = \bar{x}^T \bar{\lambda} x$$

BUT $Ax = \lambda x \Rightarrow \bar{x}^T A x = \lambda \bar{x}^T x$

SO $\lambda \bar{x}^T x = \bar{\lambda} \bar{x}^T x$ ③

$\lambda = \bar{\lambda}$, no imaginary component

③ $\bar{x}^T x$ needs to be nonzero

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \bar{x}^T = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_n \end{bmatrix} = \bar{x}_1 x_1 + \ldots + \bar{x}_n x_n$$

$$(a-ib)(a+ib) = a^2+b^2$$

imaginary parts drop out

so $\bar{x}^T x$ is real & positive & we can cancel things happily. so now we're confident that $\bar{\lambda} = \lambda$

SUPPOSE COMPLEX MATRIX $\bar{A}$. THEN $\bar{x}^T \bar{A}^T = \bar{x}^T \bar{\lambda}$.

WHAT DO WE DO HERE? OUR "GOOD" MATRICES ARE ACTUALLY $A = \bar{A}^T$ BUT FOR REAL WE IGNORE IMAGINARY PARTS. UNSATISFACTORY IMO, MAYBE MISSING SOMETHING

BUT ANYWAY FOR $A = \bar{A}^T$ $\quad A = Q \Lambda Q^T$

$$= \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} - & q_1 & - \\ & \vdots & \\ - & q_n & - \end{bmatrix}$$

$$= \lambda_1 q_1 q_1^T + \ldots + \lambda_n q_n q_n^T$$

EVERY SYMMETRIC MATRIX CAN BE BROKEN UP LIKE THIS. THIS IS A COMBINATION OF MUTUALLY PERPENDICULAR PROJECTION MATRICES!

ARE λ FOR SYMMETRIC MATRICES POSITIVE OR NEGATIVE?
WE DON'T WANT TO COMPUTE EIGENVALUES FOR $\mathbb{R}^{50}$ MATRICES WITH $|A-\lambda I|$:7 BUT WE CAN QUICKLY GET PIVOT VALUES FOR $\mathbb{R}^{50}$ MATRICES & THE SIGNS OF THE PIVOTS CORRESPOND TO THE SIGNS OF λs! # pos pivots = # pos λ. USEFUL COMPUTATIONAL TRICK!

POSITIVE DEFINITE MATRICES

- SYMMETRIC
- ALL EIGENVALUES ARE POSITIVE (AND SO ARE PIVOTS)

$$\begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix} \rightarrow \text{pivots } \{5, 11/5\}$$

$$\lambda_s: \quad \lambda^2 - 8\lambda + 11 = 0$$

$$= 4 \pm \sqrt{5}$$

- ALL SUB-DETERMINANTS ARE POSITIVE

LECTURE 26 COMPLEX MATRICES

FAST FOURIER TRANSFORM F_n mult in $O(n \log_2 n)$ instead of $O(n^2)$

WHAT HAPPENS WHEN VECTORS ARE COMPLEX?

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \text{ in } \mathbb{C}^n$$

what is its length? $z^T z$ doesn't work anymore. maybe $\bar{z}^T z$? ✓

LENGTH: $\bar{z}^T z$ ← forces positive length for nonzero vectors

$$z = \begin{bmatrix} 1 \\ i \end{bmatrix} \quad \bar{z} = \begin{bmatrix} 1 \\ -i \end{bmatrix} \Rightarrow \bar{z}^T z = 1 + 1 = 2$$

$$z^T z = 1 - 1 = 0$$

$\bar{z}^T \Rightarrow z^H$ "HERMITIAN" IS THE COMPLEX CONJUGATE TRANSPOSED

WHAT ABOUT INNER PRODUCT?

$$y^T x \Rightarrow y^H x$$

OBVIOUSLY REAL VECTORS ARE JUST SPECIAL CASES WHERE $x^T = x^H$.

SYMMETRIC $A^T = A$ ← no good if A is complex

$A^H = A$

example: $\begin{bmatrix} 2 & 3+i \\ 3-i & 5 \end{bmatrix} = A$, $A^H = \begin{bmatrix} 2 & 3+i \\ 3-i & 5 \end{bmatrix}$

HERMITIAN MATRICES ARE SYMMETRIC UNDER $A^H = A$

HAVE $\perp$ EIGENVECTORS & REAL EIGENVALUES

$$q_i^H q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \qquad Q = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}$$

$Q^T Q = I$ for real matrices

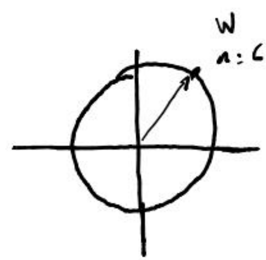
$Q^H Q = I$ for complex matrices

instead of "orthogonal" matrix we say "unitary"

THE FOURIER MATRIX

$$F_n = \begin{bmatrix} 1 & 1 & 1 & & 1 \\ 1 & w & w^2 & & w^{(n-1)} \\ 1 & w^2 & w^4 & \cdots & w^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & & w^{(n^2-2n+1)} \end{bmatrix} \qquad (F_n)_{ij} = w^{ij}$$

$i,j = 0, \ldots, n-1$



$$w^n = 1 \Rightarrow w = e^{i2\pi/N} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$

w $n=6 \rightarrow 60°$, $w^2 = 120°$ etc

$n=4 \to w=i \quad w^2=-1,\ w^3=-i,\ w^4=1$

$$F_4 = \begin{array}{c} c_1\ c_2\ c_3\ c_4 \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \end{array} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}$$

WHAT'S GOOD ABOUT THIS MATRIX? MULTIPYING APPLIES THE FOURIER TRANSFORM, AND MULTIPLYING BY F_n^{-1} UNDOES IT!

NOTE THE THE COLUMNS HERE ARE ORTHOGONAL.

WE CAN MAKE ORTHONORMAL BY DIVIDING BY 2

AND THEN $F_4^H F_4 = I$; $F_4^{-1} = F_4^H$

HOW DO WE GET TO FFT? $F^n \to F^{2n}$ LINKED BY HALVING AROUND UNIT CIRCLE;

$$\underbrace{\left[F_{64} \right]}_{64^2 \text{ calcs}} = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \underbrace{\begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix}}_{2(32)^2 + 32 \text{ calcs}} \left[P \right]$$

$\left[P \right]$ ← organising into even and odd

D = diagonal powers of w

OBV NEXT WE ATTACK F_{32}! THEN F_{16} ...

RECURSION IS OUR FRIEND

$$2\cdot\left[2[16]^2+16\right]+32$$

$$\hookrightarrow 2\left[2\left[2[8]^2+8\right]+16\right]+32 \rightarrow \log 64\cdot \tfrac{64}{2},\ \tfrac{1}{2}n\log_2 n$$

$n=2^{10}$

$n^3 \approx 10^6$

$\frac{1}{2}n\log_2 n = 512\cdot 10$:>

P splits odd and even rows because F_n is a skip-one solution for F_{2n}!

LECTURE 27: POSITIVE DEFINITE MATRICES

TESTS FOR MINIMUM

ELLIPSOIDS IN $\mathbb{R}^n$

$x^TAx>0$ is the guy to watch.

how do we tell if a matrix is positive definite? what does it mean?

$$A=\begin{bmatrix} a & b \\ b & c\end{bmatrix}$$

① $\lambda_1>0;\ \lambda_2>0$ ③ pivots >0 $\quad \frac{ac-b^2}{a}>0$

② $\underbrace{a>0}_{1\times1\ \text{det}}$ $\quad \underbrace{ac-b^2>0}_{|A|}$ ④ $x^TAx>0$

THIS ONE IS INTERESTING & IMPORTANT

EXAMPLES

$\begin{bmatrix} 2 & 6 \\ 6 & c \end{bmatrix}$ $c \geq 18$ for ③ valid (border is positive semi-def)
$\lambda_1 = 0,\ \lambda_2 > 0$

take $c = 18$?

$x^T A x$ with $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

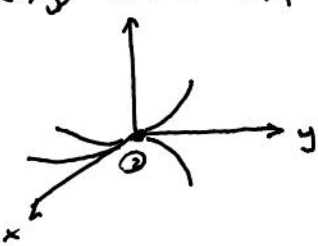
$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2x_1 + 6x_2 \\ 6x_1 + 18x_2 \end{bmatrix} = \overbrace{2x_1^2}^{ax_1^2} + \underbrace{12x_1x_2}_{2bx_1x_2} + \overbrace{18x_2^2}^{cx_2^2} \quad ①$$

QUADRATIC FORM

① IS THIS POSITIVE FOR EVERY x?

NOT FOR $c = 18$

GRAPH $f(x, y) = x^T A x = ax_1^2 + 2bx_1x_2 + cx_2^2$ take $a = 2$, $b = 6$, $c = 7$

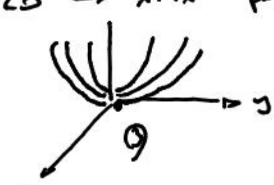


creates a saddle point - up in some directions; down in others

② saddle point - min in some directions, max in others

take $c = 20 \rightarrow x^T A x$ positive everywhere except $\{0\}$

$f(x, y) = 2x_1^2 + 12x_1x_2 + 20x_2^2$



③ MINIMUM OF A BOWL
$d' = 0$, d'' positive

FOR LINEAR ALGEBRA MIN: MATRIX OF 2^{ND} DERIVATIVES

$f(x_1, \ldots, x_n)$ IS POSITIVE DEFINITE

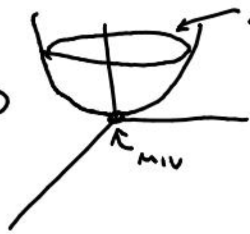
I ASSUME "DEFINITE" IS JUST SAYING THE SQUARED TERM ALWAYS BEATS THE MIXED TERM, IS THERE AN EQUIVALENT NEGATIVE?

$$\underset{1}{f(x,y) = 2x^2 + 12xy + 20y^2} = \underset{2}{\underbrace{2(x+3y)^2 + 2y^2}}$$

WE CAN MAKE THIS ALL SQUARES

$$f(x,y) = 2x^2 + 12xy + 7y^2 = 2(x+3y)^2 - 11y^2$$

cut bowl to get ellipse

1 

1 $f(x,y)$ $x=1$ $y=1$ cut

2 pivots outside, multipliers inside — $\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \to \overset{U}{\begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}}$

$\downarrow$

$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} L$

this means we know how to work in $n \times n$ matrices. why positive pivots are a test

Q: WHY DO SUBSETS MATTER?

3 $\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ det controlling

3x3 EXAMPLE

det 2   det 3

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad |A| = 4 \qquad \text{pivots: } 2,\ 3/2,\ 4/3$$

$$\lambda_s = 2-\sqrt{2},\ 2,\ 2+\sqrt{2}$$

$x^T A x = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3$ always positive for any $[x]$

take $x^T A x = 1$; equation of ellipsoid in $\mathbb{R}^3$

axes in direction of eigenvectors, length is $1/\lambda_s$

$Q \Lambda Q^{-1}$ – principle axis diagonalisation for symmetric matrices

LECTURE 28 – SIMILAR MATRICES

remember $x^T A x > 0$ is P.D. definition

comes from least squares

suppose A is positive definite. is A^{-1} also? yes –

we know $\lambda \to 1/\lambda$

if A, B are positive definite, $A+B \to$ pos def

$x^T A x > 0$; $x^T B x > 0 \Rightarrow x^T (A+B) x > 0$

NOW A is $m \times n$

A not square or symmetric but A^TA is both

is A^TA positive definite?

x^TA^TAx can't be negative because $(Ax)^T(Ax)$

$= \|Ax\|^2 \geq 0$

how do we get rid of the $x^TA^TAx = 0$ possibility?

when rank $= n$! ties straight back to linear regression

for positive definite matrices we never have to

row exchange

SIMILAR MATRICES A, B ($n \times n$, not neccesarily symmetric)

$B = M^{-1}AM$ ← means for some M, can turn $A \to B$

by multiplying left and

right by an M

EXAMPLE: $S^{-1}AS = \Lambda$! A similar to Λ

allows us to put matrices into familiés - diagonal

is simplest & easiest

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

$$\overset{M^{-1}}{\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}} \overset{M}{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 9 \\ 1 & 6 \end{bmatrix} = \overset{B}{\begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix}}$$

A, Λ, B all have something in common - eigenvalues!

$\lambda_1 = 3, \ \lambda_2 = 1$ (check via det & trace)

what does this mean in terms of matrix geometry?

$Ax = \lambda x. \qquad B = M^{-1} A M$

$AMM^{-1}x = \lambda x$

$M^{-1}AMM^{-1}x = \lambda M^{-1} x$

$BM^{-1}x = \lambda M^{-1} x \Rightarrow \lambda$ eigenvalue of B! eigenvector changes

$(\vec{\text{eig}}\, B = M^{-1}(\vec{\text{eig}}\, A))$

WHAT HAPPENS WHEN λs ARE THE SAME? BAD CASE

SUPPOSE $\lambda_1 = \lambda_2 = 4$

one family has $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ ← only one in family because $M^{-1}IM = I$

TWO FAMILIES!

another has (bigger family) $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ ①

① NEAREST TO DIAGONAL

JORDAN FORM

JORDAN FORM IS NOT EASY TO FIND FOR GENERAL MATRIX — NUMERICALLY WEIRD BUT ALGEBRAICALLY INTERESTING. SOME INTERESTING LINEAR ALGEBRA HISTORY HERE

LET'S LOOK FOR MORE IN $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ DET = 16, TRACE = 8

$$\overset{B}{\begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}}, \begin{bmatrix} 4 & 0 \\ 17 & 4 \end{bmatrix}, \begin{bmatrix} a & b \\ c & 8-a \end{bmatrix} \leftarrow \text{make det } 16, \text{ trace} = 8$$

$(8a - a^2) - bc = 16$. same λs and # of independent $\overrightarrow{\text{eig}}$

EXAMPLE

$$\left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

$\lambda_1 = 0 = \lambda_2 = \lambda_3 = \lambda_4$

Two $\overrightarrow{\text{eig}}$ $(\dim N(A) = 2)$

① change to 7; no change to λ or # $\overrightarrow{\text{eig}}$

$$J = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\lambda_1 = 0 = \lambda_2 = \lambda_3 = \lambda_4$

$\dim N(A) = 2$

BUT NOT SIMILAR!

JORDAN BLOCK J_i $\begin{bmatrix} \lambda_i & & 0 \\ & \lambda_i & \\ 0 & & \lambda_i \end{bmatrix}$

has 1 eigenvector

JORDAN'S THEOREM: EVERY SQUAR A IS SIMILAR TO JORDAN MATRIX J

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_N \end{bmatrix} \quad \text{\#blocks} = \#\vec{e_{ij}}$$

good case: J is Λ

WHY DO WE CARE?