

18.03: DIFFERENTIAL EQUATIONS

LECTURE 1: The Geometrical view of $y' = f(x, y)$

First-order ODEs: $y' = f(x, y)$

Examples: $y' = x/y$; $y' = x - y^2$; $y' = y - x^2$

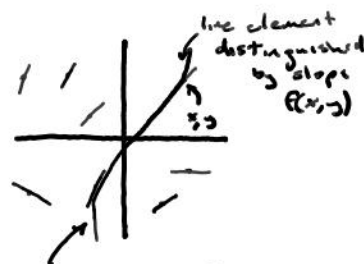
$y' = x/y$ is solvable by separating variables
 $y' = x - y^2$ has no elementary functions to help solve "recalcitrant"
 $y' = y - x^2$ is easily solvable

Most equations are unpleasant! Let's look at a zoomed-out view before we start trying to solve anything with the analytic method $\rightarrow y' = f(x, y)$
 $y_1(x)$ solution

THE GEOMETRIC VIEW OF ODES

Direction field $\hat{=}$ integral curve

$y' = f(x, y)$ $\leftrightarrow$ graph of solution



integral curve has direction of field everywhere

$y_1(x)$ solution to $y' = f(x, y)$

if and only if the graph $y_1(x)$

is an integral curve of direction field associated with that equation

$y_1'(x) = f(x, y_1(x)) \iff$ slope of $y_1'(x)$ = slope of d.f. $f(x, y_1(x))$ at $(x, y_1(x))$

ANALYTICAL

GEOMETRIC

Drawing a direction field

Computer method: pick a point (x, y) [equally spaced]

computes $f(x, y)$

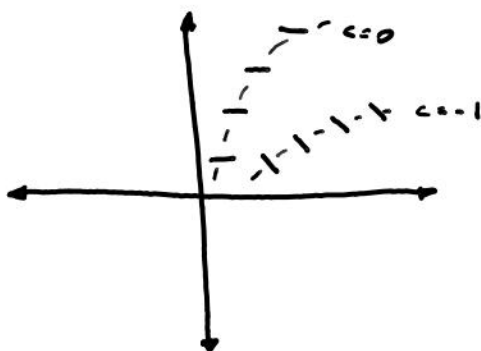
draw line element with slope $f(x, y)$

human method:

pick slope c

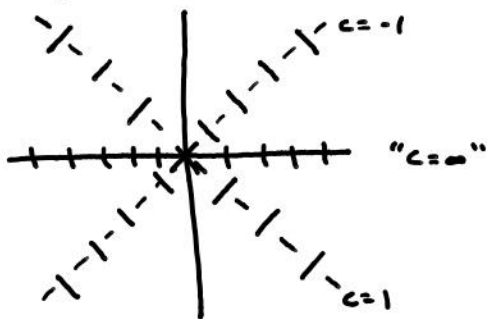
plot $f(x, y) = c \leftarrow$ isocline

draw line elements with slope c
on isocline



① rewrite $y' = f(x)g(y)$
as $\frac{dy}{g(y)} = f(x)dx$; int. both s.d.s,
then solve for y .

Example: $y' = -x/y$ $-x/y = c \Rightarrow y = \frac{-1}{c}x$

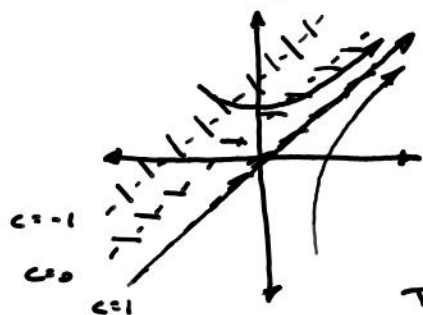


integral curves everywhere tangent to line segments:

they're circles: $x^2 + y^2 = c_1^2$. solve by separating variables

$y = y(x) = \sqrt{c_1^2 - x^2}$ ← limited domain solution

Example: $y' = 1 + x - y$



isocline: $y = x + 1 - c$

corridor between $c=0$ & $c=2$ isos
no escape!



TWO INTEGRAL CURVES CANNOT CROSS

BECAUSE LINE ELEMENT HAS DEFINED SLOPE AT A POINT!

In this case the integral curve at the $c=1$ isocline is asymptotic! $\therefore y(x) \sim x$ is a solution found solely from geometry!

TWO INTEGRAL CURVES CANNOT BE TANGENT No!

REASON: Existence and uniqueness theorem

existence

(x_0, y_0)

$y' = f(x, y)$ has

HYPOTHESIS: $f(x, y)$ continuous near (x_0, y_0)

& $\partial/\partial y f(x, y)$ continuous near (x_0, y_0)

one & only one solution through (x_0, y_0)

uniqueness

EXAMPLE: $xy' = y - 1$

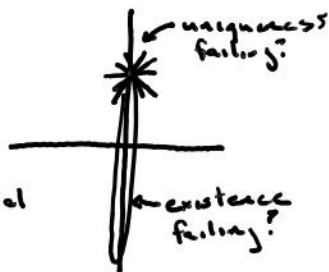
$$\frac{dy}{y-1} = \frac{dx}{x} \Rightarrow \ln|y-1| = \ln|x| + C$$

$$y-1 = Cx$$

reason this is a mess is because actual

form is $dy/dx = (y-1)/x$

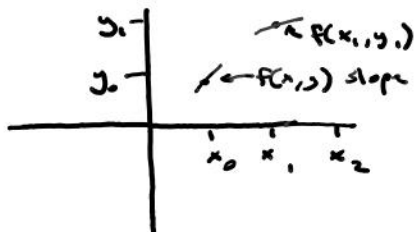
not defined along $x=0$



LECTURE 2: NUMERICAL METHODS, EULER'S METHOD

$$\left. \begin{array}{l} y' = f(x, y) \\ y(x_0) = y_0 \end{array} \right\} \text{initial value problem}$$

EULER'S METHOD: General numerical solution



choose uniform step size h

$$x_0 + h = x_1, \text{ etc.}$$

$$y_{n+1} - y_n = h A_n$$

$$y_{n+1} = h A_n + y_n$$

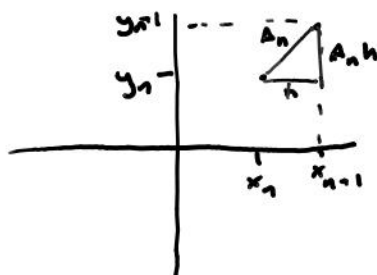
$\Downarrow$

Euler equations

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + h A_n$$

$$A_n = f(x_n, y_n)$$



Example: $y' = x^2 - y^2$

$$y(0) = 1$$

$$h = 0.1$$

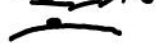
n	x_n	y_n	A_n	$h A_n$
0	0	1	-1	-0.1
1	0.1	0.9	-0.8	-0.08
2	0.2	<u>0.82</u>		

stop @ $y(0.2) = 0.82$

how right is this? is Euler too high or too low?

we can answer this geometrically.

if solution is straight line, Euler's method is accurate. if curve...

$y_1(x)$  convex - always low
 concave - always high

but what if we don't know what the solution looks like? use y'' : $y'' < 0$ for concave; $y'' > 0$ convex

$$y' = x^2 - y^2$$

$$y'' = 2x - 2yy'$$

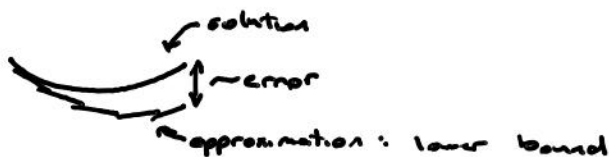
@ (0,1) $y(0) = 1$

$$y'(0) = -1$$

$$y''(0) = 2 \cdot 0 - 2 \cdot 1 \cdot (-1) = 2$$

$\therefore$ solution convex; approx is low

for convex...



but how do we do better?

1) take smaller steps $h = 0.1 \Rightarrow h = 0.01$ smaller error

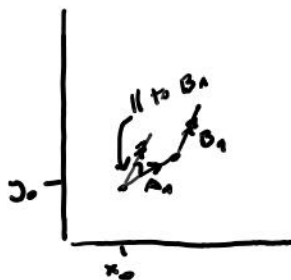
(surely an infinitesimal step size works)

$e \sim c_1 h$ ← Euler is a first-order method $[h^1]$



2) try to improve An^-

take next step, average with first, that's your corrected slope.



$$x_{n+1} = x_n + h$$

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + h(A_n + B_n)/2$$

temporary value $\tilde{y}_{n+1}$

$$B_n = \text{slope of direction field } f(x_{n+1}, \tilde{y}_{n+1})$$

Heun's method / improved Euler / modified Euler / RK2^①

① "2" because it's a second-order method; i.e. error decreases with the square of the step size

(there is an RK4; standard method in computational analysis. $(A_n + 2B_n + 2C_n + D_n)/6$ for slope

Pitfalls

13 We'll find on our own!

$$2) \quad y' = y^2$$

solutions: $y = \frac{1}{c-x}$

singularity at $x=c$. but you can't tell
where c is in advance!



asymptotes are
upsetting

LECTURE 3: Solving First-order ODEs

Two most important methods: separation of variables & first-order linear equations. Today's lecture is about the latter

$$\underbrace{a(x)y' + b(x)y = c(x)}_{\text{"first-order linear" - linear in } y \text{ \& } y'} \leftarrow y \text{ is limited; } x \text{ essentially arbitrary}$$

if $c(x) = 0$, we call the system homogeneous

Standard linear form

$$\textcircled{1} y' + p(x)y = q(x) \leftarrow \text{normalise } y' \text{ coefficient, so } b(x)/a(x) \Rightarrow p(x) \\ \& \ c(x)/a(x) \Rightarrow q(x)$$

standard first-order form

$$y' = p(x)y + q(x) \leftarrow p(x) \text{ here is } -1 \cdot p(x) \text{ in standard linear}$$

we will avoid the standard first-order form in this course

① can always be solved, and also arises often in models

• Temperature-concentration model ②

• Mixing model

• Radioactive decay

② CONDUCTION (TEMP)

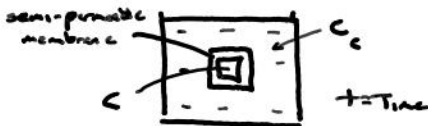


use Newton's law of cooling
to build ODE.

$$\frac{dT}{dt} = k(T_c - T) \leftarrow \text{try to keep } k > 0!$$

take $T(0) = T_0$

DIFFUSION (CONCENTRATION)



$$\frac{dC}{dt} = k(C_c - C) \text{ for } C_c > C$$

Convert these to standard linear form

$$\frac{dT}{dt} + kT = kT_c \quad \leftarrow \text{general} \quad / \quad \frac{dC}{dt} + kC = kC_c$$

$\uparrow$
usually constant but not necessarily

$$\boxed{y' + p(x)y = q(x)}$$

we solve this equation by finding an integrating factor of form $u(x)$.

we multiply everything through by u

$uy' + puy = qu$, where $uy' + puy$ turns out to be the derivative of something wrt x . then we integrate both sides wrt x .

How do we pick u ? In this case we would like $uy' + puy$ to turn into $(uy)'$. u must satisfy $u' = p \cdot u$

separate variables...

$$\ln u = \int p(x) dx$$

$$u = e^{\int p(x) dx} \leftarrow \text{no arbitrary constant since only one } u \text{ required}$$

$$\text{Method: } y' + py = q$$

- ① Force standard linear form to ensure correct p
- ② Find integrating factor $e^{\int p dx} = u$
- ③ Multiply both sides by u (don't forget right hand side)
- ④ Integrate

Example: $xy' - y = x^3$

① $y' - \frac{1}{x}y = x^2$

② i.f. = $e^{-\int \frac{1}{x} dx} = (e^{\ln x})^{-1} = 1/x$

③ $\frac{1}{x}y' - \frac{1}{x^2}y = \frac{x^2}{x}$

④ $(\frac{1}{x}y)' = x$ helpful for checking!

$\frac{1}{x}y = x^2/2 + c \rightarrow y = \underline{\underline{x^3/2 + cx}}$ ← solution

Example: $(1 + \cos x)y' - (\sin x)y = 2x$

① $y' - [\sin(x)/(1 + \cos x)]y = 2x/(1 + \cos x)$

② i.f. $\Rightarrow -\int \frac{\sin x}{1 + \cos x} dx = \ln(1 + \cos x) \Rightarrow$ i.f. = $e^{\ln(1 + \cos x)} = 1 + \cos x$

③ $(1 + \cos x)y' - (\sin x)y = 2x$

④ $[(1 + \cos x)y]' = 2x$

$(1 + \cos x)y = x^2 + c \Rightarrow y = (x^2 + c)/(1 + \cos x)$

if $y(0) = 1$, $1 = c/2 \Rightarrow c = 2$ $\therefore$ full solution is $\frac{x^2 + 2}{1 + \cos x}$

Example: Linear with k constant

$dT/dt + kT = kT_c$

i.f. = e^{kt}

$(e^{kt}T)' = kT_c \cdot e^{kt}$

$e^{kt}T = \int kT_c(t) e^{kt} dt + c \Rightarrow T = e^{-kt} \int kT_c e^{kt} dt + c e^{-kt}$ $\rightarrow 0$ as $t \rightarrow \infty$

but definite solution preferred $\Rightarrow \underbrace{e^{-kt} \int_0^t kT_c(\tau) e^{k\tau} d\tau}_{\text{steady-state solution}} + c e^{-kt}$; $c \leq T_0$

SIDE QUEST: Separation of variables

To apply separation of variables in a first-order ODE, we rewrite the system such that the y-side depends only on y & dy and the x-side only on x & dx

$$\text{e.g. } y' = f(x)g(y) \Rightarrow dy/g(y) = f(x)dx$$

as long as the terms don't mix (i.e. no $x+y$ terms) we can then integrate both sides & solve

$$\text{EXAMPLE: } y' = (x-y)/x; \quad x \neq 0 \\ = 1 + y/x$$

$$\text{set } v = y/x; \quad y = vx$$

$$y' = v + x \, dv/dx$$

$$v + x \, dv/dx = 1 + v$$

$$x \, dv/dx = 1 \Rightarrow \frac{dv}{dx} = \frac{1}{x}$$

$$v = \ln|x| + C \Rightarrow y = x(\ln|x| + C)$$

NOTE THAT WE HAD TO SUBSTITUTE FOR SEPARATION TO EVEN BE POSSIBLE - JUST HAVE TO RECOGNISE PATTERNS & FAMILIES.

BE CAREFUL WHEN POSSIBLE FOR $g(y) = 0$!

LECTURE 4: First-order substitution methods

We know how to solve two kinds of first-order differential equations: 1) separable 2) linear. These are the only two kinds of differential equations with general solutions - to a great extent all other solvable equations are reducible to one of these two cases, so substitution is important.

Scaling: $y' = f(x, y)$

$$x_1 = x/a, \quad y_1 = y/b \quad \text{with } a, b \text{ constant}$$

we do this to e.g. change units, make variables dimensionless, or reduce the number or simplify the constants in an equation.

Scaling Example

$$\frac{dT}{dt} = k(M^4 - T^4) \quad \text{where } T \text{ is internal temp } \hat{=} M \\ \text{is external constant; applies for large } \Delta T$$

← dimensionless ΔT

scale: $T_1 = T/M \Rightarrow T = MT_1$

$$\frac{dT_1}{dt} \cdot M = kM^4(1 - T_1^4) \Rightarrow \frac{dT_1}{dt} = \overbrace{kM}^{\text{call this } k_1, \text{ units are } \cancel{Y}/t} (1 - T_1^4)$$
$$= k_1(1 - T_1^4)$$

Direct substitutions $\rightarrow$ new variable equal to some combination of old variables ($T_1 = T/M$)

Inverse substitutions $\rightarrow$ old variables are some combination of the new variables ($T = MT_1$)

$$\int x\sqrt{1-x^2} dx ; u=1-x^2 \quad (\text{direct})$$

$$\int \sqrt{1-x^2} dx ; x = \sin u \quad (\text{indirect})$$

Example: Direct substitution

$$y' = \underbrace{p(x)y + q(x)}_{\text{Bernoulli equation (no pure } x \text{ term)}} y^n \quad (n \neq 0 \text{ or } 1)$$

Bernoulli equation (no pure x term)

Approach: divide through by y^n

$$y'/y^n = p(x) y^{n-1} + q(x); \quad \text{let } v = y^{n-1}$$

$$v' = (1-n) y^{n-2} \cdot y'$$

$$\Rightarrow v'/1-n = p(x) \cdot v + q(x) \quad \text{linear!}$$

now we can solve using our linear approach & back-substitute

$$\text{Example: } y' = \frac{y}{x} - y^2 ; \quad v = \frac{1}{y} ; \quad v' = -\frac{1}{y^2} \cdot y'$$

$$y'/y^2 = \frac{1}{x} y^{-1}$$

$$-v' = v/x - 1 \rightarrow v' + v/x = 1$$

$$\text{integrating factor: } e^{\int \frac{1}{x} dx} = x$$

$$xv' + v = x \Rightarrow (xv)' = x$$

$$v = \frac{1}{2}x + C/x \rightarrow \text{BACK SUBSTITUTE} \rightarrow \frac{2x}{x^2+C} = y$$

Homogeneous first-order ODEs

$y' = F(y/x)$ ← homogeneous means being able to write right-hand side as the function of combined variable y/x

$$y' = x^2 y / (x^3 + y^3) = (y/x) / (1 + (y/x)^3) \quad \text{homogeneous}$$

$$xy' = \sqrt{x^2 + y^2} \Rightarrow \sqrt{1 + y^2/x^2} = y$$

homogeneous equations are invariant under zoom (b/c

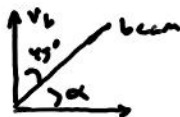
y/x is a ratio!) $x \rightarrow ax, y \rightarrow ay$.

$$dy/dx = a dy/adx = dy/dv = F(y/x)$$

$$y' = F(y/x) \quad z = y/x \Rightarrow xz = y \Rightarrow y' = z'x + z$$

$$z'x + z = f(z) \Rightarrow x dz/dx = F(z) - z; \text{ separate variables}$$

& back-substitute



what's the boat's path? want a curve — we're looking for unknown function $y = y(x)$ whose slope makes a known angle

$$y' = \tan(\alpha + 45^\circ) ; \tan \alpha = y/x$$

$$= (\tan \alpha + \tan 45^\circ) / (1 - \tan \alpha \tan 45^\circ)$$

$$y' = \frac{(y/x + 1)/(1 - y/x)}{x - y} = \frac{y+x}{x-y} ; \text{ let } z = y/x ; y' = z'x + z$$

$$x \frac{dz}{dx} = \frac{z+1}{1-z} - z = \frac{1+z^2}{1-z}$$

$$\frac{1-z}{1+z^2} dz = \frac{dx}{x} \Rightarrow \tan^{-1} z - \ln(1+z^2)/2 = \ln x + C$$

$$\tan^{-1} z = \ln \sqrt{1+z^2} + \ln x + C ; z = y/x \Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \ln \sqrt{x^2 + y^2} + C$$

exponential spiral $0 = \ln r + C \Rightarrow r = C e^0$

PROBLEM SET 1:

1) A mysterious virus infects the oryxes of the Tana River area in Kenya, which causes their growth rate to decrease according to the formula $k(t) = k_0 / (a+t)^2$ for $t \geq 0$, where $a \in \mathbb{R}$, k_0 are positive constants. Measure time in years t ; population in kilo-oryxes (k_0). Without the virus the $\dot{x}$ for small time intervals Δt , the population change $x(t+\Delta t) - x(t)$ is well approximated by $kx(t)\Delta t$.

a) What are the units of the constant a in " $a+t$ ", and of the constant k_0 ?

$x(t)$ is in years⁻¹, so k_0 in years $\dot{x}$ & a in years works

b) Write down the differential equation modelling this situation

Replace original k with $k(t)$, so:

$$\frac{x(t+\Delta t) - x(t)}{\Delta t} \approx x(t) + k_0 / (a+t)^2 x(t) \Delta t \Rightarrow \frac{dx}{dt} = \frac{k_0 x}{(a+t)^2}$$

c) Write down the general solution to this differential equation. Separate variables $\Rightarrow dx/x = k_0 dt / (a+t)^2$; integrate so $\ln|x| + C_1 = -k_0 (a+t)^{-1} + C_2$

$$\ln|x| = -k_0 (a+t)^{-1} + C$$

$$|x| = e^{C - k_0 (a+t)^{-1}} ; \text{ let } C = \pm e^C \parallel 0$$

$$x = C e^{k_0 (a+t)^{-1}}$$

1 cont) d) Now suppose that at $t=0$ there is a positive population x_0 of oryx. Does the progressive decline in growth rate cause the population to stabilise for large time, or does it grow without bound? If it does stabilise, what is the limiting population as $t \rightarrow \infty$?

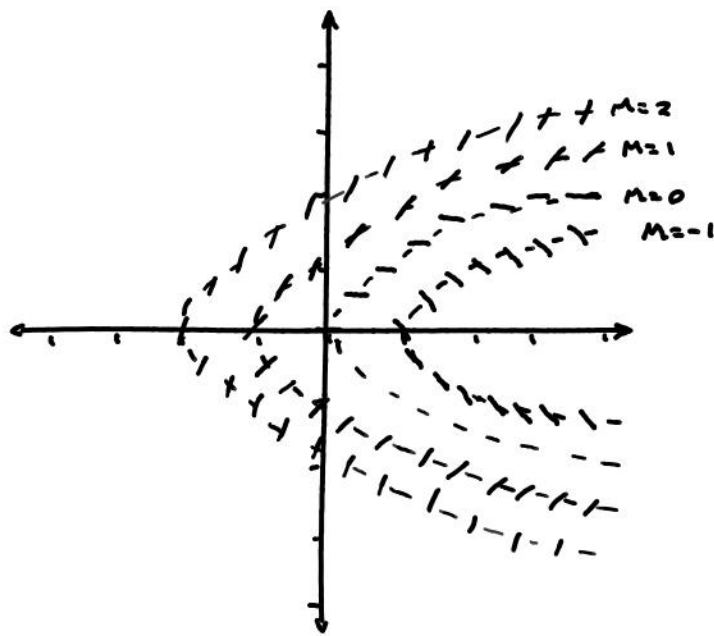
$$x = C e^{-k_0(a+t)^{-1}} \quad \text{As } t \rightarrow \infty \text{ drives } e^{\wedge} \text{ term to } 1, \therefore x_{t \rightarrow \infty} = C$$

$$x(0) = x_{\infty} e^{-k_0/a} \Rightarrow x_{\infty} = e^{-k_0/a} / x_0$$

2) In this problem you will study the solutions of the differential equation $dy/dx = y^2 - x$. Solutions of this equation do not admit expressions in terms of the standard functions of calculus, but we can study them anyway using the direction field. Draw the field & isoclines from $x, y \in [-4, 4]$

$$M = -1, 0, 1, 2$$

$$y^2 - x = M$$



3) a) write y for the solution to $y' = 2x$ with $y(0) = 0$. What is $y(1)$? What is the Euler approximation for $y(1)$, using 2 equal steps? 3 equal steps? What about $n \in \mathbb{N}$ steps? As $n \rightarrow \infty$, these approximations should converge to $y(1)$. Do they?

$$y = x^2 \text{ by inspection} \Rightarrow y_1 = 1$$

for $h = 1/2$

n	x_n	y_n	Δ_n	$h\Delta_n$
0	0	0	0	0
1	0.5	0	1	0.5
2	1	0.6	-	-

for $h = 1/3$

n	x_n	y_n	Δ_n	$h\Delta_n$
0	0	0	0	0
1	$1/3$	0	$2/3$	$2/9$
2	$2/3$	$2/9$	$4/3$	$4/9$
3	1	$2/3$	-	-

for $h = 1/n$

n	x_n	y_n	Δ_n	$h\Delta_n$
0	0	0	0	0
1	$1/n$	0	$2/n$	$2/n^2$
2	$2/n$	$2/n^2$	$4/n$	$4/n^2$
3	$3/n$	$4/n^2$	$6/n$	$6/n^2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$y_n = n(n-1)/n^2 = (n-1)/n \quad \lim_{n \rightarrow \infty} [(n-1)/n] = \lim_{n \rightarrow \infty} (1 - \frac{1}{n}) = 1$$

$\therefore$ approximations converge

3 cont) b) What is the exact value of the difference between $y(1)$ & the Euler approximation! $\boxed{1/n}$ or h !

4) Scrooge McDuck wants to set up a trust fund for his nephew Don. He has fool-proof investments which make a constant interest rate of, measured in (years)⁻¹, and he proposes to dole out the money to his nephew at a rate of q dollars per year.

a) Model this process by a differential equation

$$y' = Iy - q \quad \leftarrow \text{we take \$ in trust fund as } y$$

and see the changes $+ Iy \text{ t/year} +$
 $q \text{ t/year} -$

b) Find the general solution to this equation

$$\frac{dy}{dt} = Iy - q$$
$$\frac{dy}{(Iy - q)} = dt$$

$$\ln |Iy - q| = It + c$$

$$Iy - q = Ce^{It} \quad (C = \pm e^c \neq 0)$$

$$y = (Ce^{It} + q)/I$$

c) Now take $I = 0.05$. If scrooge wants q to be \$1000 per month forever, how much should he invest?

$$Iy_0 = 12,000 \Rightarrow \$240,000$$

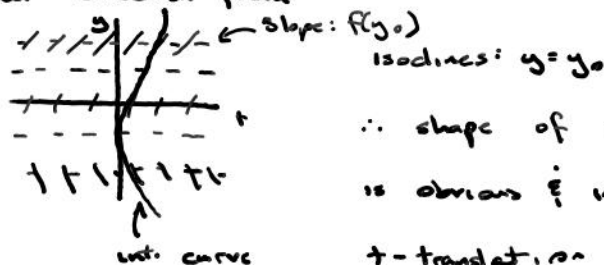
LECTURE 5: First-order autonomous ODEs

$$dy/dt = f(y)$$

special kind of ODE where right hand side has no independent variables. name is "autonomous". Can separate variables - but we can get useful information out without solving.

Problem: get qualitative info about solutions w/o actually solving

1) Look at direction field



$\therefore$ shape of integral curve is obvious & invariant under t -translation

2) Look for "critical point": zeroes of RHS

$f(y_0) = 0$ or $y = y_0$ is a solution to equation

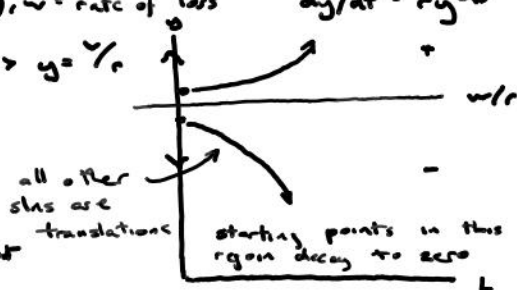
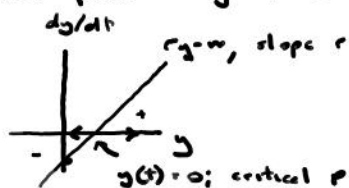
if y_0 & y_1 are solutions, they're integral curves & other integral curves cannot cross!

3) Draw graph of $f(y) > 0 < 0$? $\frac{dy}{dt} = f(y)$ — positive when y is increasing

EX: y = money; r = interest (cont), w = rate of loss

$$dy/dt = ry - w$$

critical point: $ry - w = 0 \Rightarrow y = w/r$



Logistic Equation

population behavior $y(t)$

$\frac{dy}{dt} = ky$ k = growth rate $\rightarrow k$ constant $\Rightarrow$ simple growth

Logistic growth: k declines as population increases (resources used up etc). so k wants to be a function.

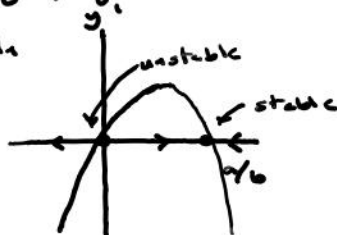
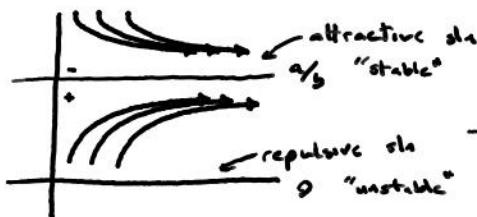
simplest: $a - by = k$

$\frac{dy}{dt} = ay - by^2 \leftarrow$ logistic equation; many applications

we can avoid partial fractions & other unpleasantness

by not solving. 1)

CRITICAL POINTS: $y(a - by) \Rightarrow y = 0, y = a/b$



so we understand stable & unstable critical points.

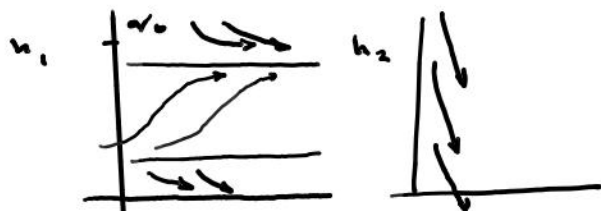
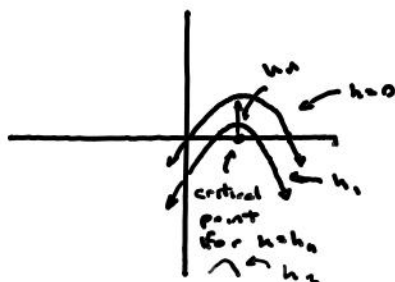
what about the third case?

stable from one side, unstable

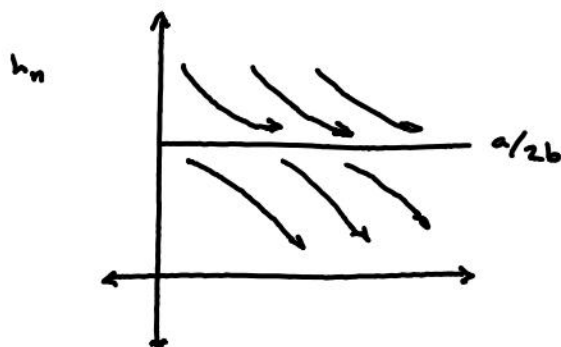
from another $\rightarrow$ "semi-stable"



Logistic Equation with harvesting
 harvest at constant rate $\frac{dy}{dt} = ay - by^2 - h$



h values vertically
 translate parabola



h_m is max rate
 of harvesting without
 extinction

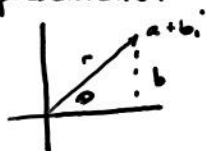
LECTURE 6: Complex numbers & complex exponentials

$$z = a + bi \quad \bar{z} = a - bi \quad \underbrace{z\bar{z}}_{\text{real}} = a^2 + b^2$$

$(2+i)/(1-3i)$ $\downarrow$ mult by cc of denominator

$$\frac{2+i}{1-3i} \cdot \frac{1+3i}{1+3i} = \frac{-1+7i}{10} = -1/10 + 7/10i$$

Polar representation



$$a + bi = r \cos \theta + i r \sin \theta = r (\cos \theta + i \sin \theta)$$

Euler def: $r \cdot e^{i\theta}$

How is $e^{i\theta}$ an exponential? What does it mean?

① Exponential law: $e^x e^y = e^{x+y}$

② Satisfies exponential law, but more importantly...

$$e^{at} dy/dt = ae^{at} \quad y(0)=1$$

To force something into exponential ① & ② should work

$$① e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}?$$

$$② d/d\theta e^{i\theta} = i e^{i\theta}?$$

Also the Taylor expansion should be right $\Rightarrow$

$$① (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2)$$

$$\underbrace{\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2}_{\cos(\theta_1 + \theta_2)} + i \underbrace{(\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2)}_{\sin(\theta_1 + \theta_2)}$$

These trig identities are equivalent to $e^{i\theta}$ satisfying the exponential law!

$$② e^{i\theta} = \cos \theta + i \sin \theta$$

$e^{i\theta}$ ← complex valued function of real variable
input $\rightarrow$ complex output

such a cvf might look like

$u(t) + i v(t)$. how do we differentiate such a thing?
by ... partials?

$$D(u + iv) = Du + iDv$$

$$d/dt e^{it} = d/dt (\cos t + i \sin t) = (-\sin t + i \cos t)$$

$$= i(\cos t + i \sin t) = i e^{it}$$

$$e^{i0} = \cos 0 + i \sin 0 = 1$$

$$\text{so } e^{a+ib} = e^a \cdot e^{ib}$$

Polar form: $re^{i\theta}$ where r is $\sqrt{a^2+b^2}$ (modulus)

θ is from $a = r \cos \theta$ (argument)
 $b = r \sin \theta$

Advantage of Polar form: good for multiplication

$$r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

i.e. multiply moduli & add the arguments

makes geometric consequence of ^{complex} multiplication clear

$\int e^{-x} \cos x \, dx$ requires integration by parts twice and is messy

instead use complex numbers

$$e^{-x} \cos x \text{ is real part of } e^{-x+ix} = \Re(e^{x(i-1)})$$

$$\therefore \int e^{-x} \cos x \, dx = \Re \int e^{x(i-1)} \, dx \leftarrow \text{complexifying the integral}$$

$$\int e^{x(i-1)} \, dx = e^{x(i-1)} / (i-1)$$

extract real part by ...

$$\begin{aligned} \Re(-1-i)^{-1} e^{-x} (\cos x + i \sin x) &= \Re(-1-i)^{-1} e^{-x} (\cos x + i \sin x) / 2 \\ &= e^{-x} / 2 \cdot (-\cos x + \sin x) \end{aligned}$$

Extraction of complex roots $\sqrt[n]{1} \Rightarrow n$ answers as complex numbers

$$n=5 \rightarrow \zeta = e^{i2\pi/5} \quad (r=1)$$



$x = 2\pi/5 \leftarrow 5 \text{ 5th roots}$

$$\zeta^5 = e^{i2\pi/5 \cdot 5} = e^{i2\pi} = 1$$

applies generally, not just for 1 on unit circle

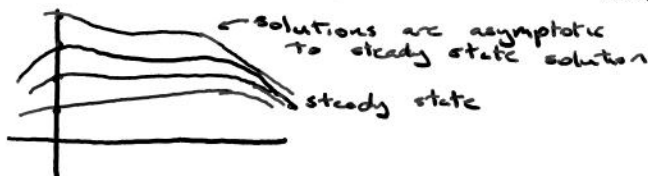
LECTURE 7: First-order linear with constant coefficients

$$y' + p(t)y = q(t) \leftarrow \text{"good" form}$$

$$\textcircled{1} y' + ky = q(t) \leftarrow \text{const coeff} \quad k > 0$$

special form: $y' + ky = kq_e(t)$ - temp/conc/mixing

$$\text{solve } \textcircled{2}: y = \underbrace{e^{-kt} \int q(t) e^{kt} dt}_{\text{steady-state}} + ce^{-kt} \rightarrow 0 \text{ as } t \rightarrow \infty; \text{ transient uses } y(0)$$



But there is no particular/privileged steady-state solution; just pick the simplest function for our purposes.



special form often written as $1/k y' + y = q_e(t)$ but we won't use it.

$\textcircled{3}$ normally any steady-state solution. Note again $k > 0$, or nothing makes any sense. Superposition of inputs: $q_1(t) \rightarrow y_1(t)$
 $q_2(t) \rightarrow y_2(t)$

what happens when the physical inputs are trigonometric?

statement of linearity of ODE

$$\left\{ \begin{array}{l} q_1 + q_2 \rightarrow y_1 + y_2 \\ c q_1 \rightarrow c y_1 \end{array} \right.$$

$$y' + ky = k q_c(t)$$

$$q_c = \cos \omega t$$

ω = angular frequency
of complete oscillations
in 2π on t -axis

Problem: For physical input $q_c = \cos \omega t$, find the response

We'll complexify the problem (since easier to integrate exps)

$$e^{i\omega t} = \cos \omega t + i \sin \omega t$$

$$\tilde{y}' + k \tilde{y} = e^{i\omega t} \quad (\text{complexified})$$

but y is real function, so
we'll call it $\tilde{y}$ from now on

$$\tilde{y} = y + i y_2 \quad (\text{complex solution})$$

Find $\tilde{y}$, then y , will solve original problem

integrating factor: e^{kt}

$$(\tilde{y} e^{kt})' = k e^{kt}$$

$$\tilde{y} e^{kt} = \frac{k}{k+i\omega} e^{(k+i\omega)t}$$

cancel

$$\tilde{y} = \frac{1}{1+i(\omega/k)} e^{i\omega t} \quad \leftarrow \text{take real part}$$

Two methods

① POLAR

② CARTESIAN

Reminder: α complex; $\frac{1}{\alpha} \cdot \alpha = 1$

$$\therefore \left| \frac{1}{\alpha} \right| = \frac{1}{|\alpha|}$$

① Polar form for $1/(1+i(\omega/k))$



$$= A e^{-i\phi}; \quad A = \frac{1}{\sqrt{1+(\omega/k)^2}}$$

$$\tilde{y} = A e^{i\omega t - i\phi}$$

$$= \frac{1}{\sqrt{1+(\omega/k)^2}} e^{i(\omega t - \phi)}$$

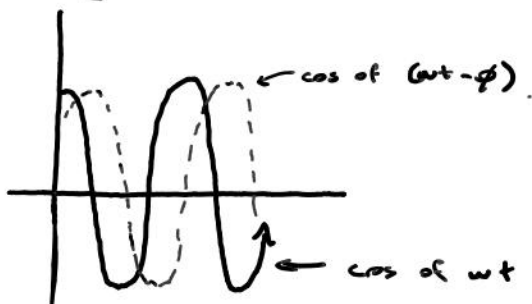
$$y = A \cdot \cos(\omega t - \phi)$$

$\phi = \tan^{-1}(\omega/k)$
phase lag of
function

$$\arg(\frac{1}{\alpha}) = \arg \alpha = 0$$

$$\therefore \arg(\frac{1}{\alpha}) = -\arg(\alpha)$$

Phase lag:



eg $\sin wt = \cos (wt - \pi/2)$

Response function

$$\underbrace{\frac{1}{\sqrt{1+(\gamma/k)^2}}}_A \cos(\omega t - \phi) \quad k \uparrow \leftarrow \text{how does this affect amplitude \& phase lag?}$$

$A \uparrow$ as $k \uparrow$ $\phi \downarrow$ as $k \uparrow$

LECTURE 3: CONTINUATION

$y' + ky = k \cos \omega t$ (we did this in L7)

$\tilde{y}' + k\tilde{y} = k e^{i\omega t}$

$\tilde{y} = \frac{1}{1+i(\gamma/k)} e^{i\omega t} \Rightarrow \underbrace{\frac{1}{\sqrt{1+(\gamma/k)^2}}}_{\text{real component}} \cos(\omega t - \phi) \quad \textcircled{D}$

Now let's try with Cartesian coordinates

$\tilde{y} = \frac{1}{1+i(\gamma/k)} e^{i\omega t}$

$= \frac{1-i(\gamma/k)}{1+(\gamma/k)^2} \cos \omega t + i \sin \omega t$

$\text{Re}(\tilde{y}) = \frac{1}{1+(\gamma/k)^2} (\cos \omega t + \gamma/k \sin \omega t) \quad \textcircled{D}$

$\textcircled{D}$ & $\textcircled{D}$ are equivalent. we can turn one into another by...

③

$$a \cos \theta + b \sin \theta = C \cos(\theta - \phi)$$



$$\text{where } C = \sqrt{a^2 + b^2}$$

$$\phi = \tan^{-1}(b/a)$$

$$\text{so: } \frac{1}{1 + (w/k)^2} \cdot \sqrt{1 + (w/k)^2} \cdot \cos(wt - \phi)$$

$$\frac{1}{\sqrt{1 + (w/k)^2}} \quad \text{so } ① = ②$$

Obviously polar coordinates are a more sensible way to look at trig terms!

Proof of ③: we need to start with $\underbrace{a \cos \theta + b \sin \theta}_{\langle a, b \rangle \cdot \langle \cos \theta, \sin \theta \rangle}$



$$= |\langle a, b \rangle| \cdot \cos(\theta - \phi)$$

Another proof: complex numbers

$$\begin{aligned} \Re[(a - bi)(\cos \theta + i \sin \theta)] &= a \cos \theta + b \sin \theta \\ &= \sqrt{a^2 + b^2} e^{-i\phi} e^{i\theta} \\ &= \sqrt{a^2 + b^2} e^{i(\theta - \phi)}; \text{ take real parts of this and} \\ &\quad e^{i(\theta - \phi)} = \cos(\theta - \phi) \end{aligned}$$

Basic Linear Equations

1. $y' + ky = k q_c(t)$ ← special, governs conduction/diffusion
2. $y' + ky = q(t)$ ← no constant on right side
3. $y' + p(t)y = q(t)$ ← modifier of y varies with T

Between cases 1 & 2 there's a very important distinction: positive k !

Mixing example:



$x(t)$ = amount of salt in tank at time t

C_e = concentration of incoming flow

r = flow rate

$$\frac{dx}{dt} = \text{rate of salt inflow} - \text{rate of salt outflow}$$

$$= r C_e - r \frac{x}{V}$$

$$\frac{dx}{dt} + r \frac{x}{V} = r C_e(t)$$

$$V \frac{dc}{dt} + r C = r C_e(t)$$

x/V
① obviously concentration, but dx/dt more subtle:

$$C_e = x/V; \quad x = C_e V, \quad dx/dt = V \frac{dC_e}{dt}$$

standard form:

$$\frac{dC}{dt} + \frac{r}{V} C = \frac{r}{V} C_e(t) \leftarrow \text{this is our temp/diff special form!}$$

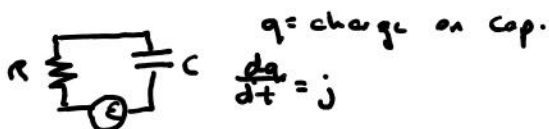
$k = r/V$ is the basic parameter governing mixing - the fractional rate of outflow

If sinusoidal C_e cos wt, how closely does concentration in tank follow $C_e(t)$.

- if k large, follows closely
- if k small, largely decoupled from oscillation

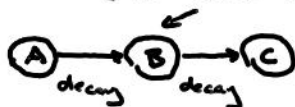
When doesn't our special case work?

How does current flow
in this circuit?



$$R \frac{dq}{dt} + \frac{q}{C} = \varepsilon(t) \rightarrow q' + q/rc = \varepsilon/r \leftarrow k \text{ not really on right side}$$

charged decay how much is θ given time?



$$dB/dt = k_1 A - k_2 B$$

$$B' + k_2 B = k_1 A_0 e^{-k_1 t}$$

$k_1 A$ has no
relationship with k_2

If $k < 0$, none of terminology of

"transient", "steady-state", "input", "response" applies. Mathematical
approach is identical, but none of our standard physical
interpretation applies.

$$dy/dt - ay = q(t) \Rightarrow k = -a \quad e^{at} \int q(t) e^{-at} dt + c e^{at}$$

initial conditions dominate!

goes to ∞ as $t \rightarrow \infty$

$k < 0$ in biology, economics, etc

Problem Set 2

4) Almost all the radon in the world today was
created in the past week or so by a chain of radioactive
decays beginning nearly from uranium. This cascade of
decaying elements is quite common, and we'll look
at a "toy" model here in which the numbers
turn out decently

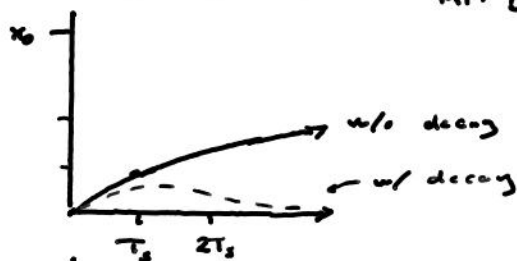
4 cont) A certain isotope of Startium, symbol St, decays with a half-life t_s . It decays with equal probability into Medium, M, & Endium, En. Medium is also radioactive, and decays with half-life t_m into Endium. Also, $t_m \neq t_s$

Use the notation $x(t)$, $y(t)$ & $z(t)$ for the amount of startium, medium & endium respectively, in units so $x(0)=1$.

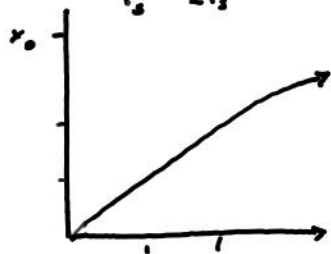
- c) Make rough sketches of graphs of x , y , z as functions of t . What are the limiting values as $t \rightarrow \infty$?



St: 0 when $t \rightarrow \infty$



M: 0 when $t \rightarrow \infty$



En: x_0 when $t \rightarrow \infty$

4) cont) b) Write down the differential equations controlling $x, y, \dot{x}$. Use σ for the decay constant for St & μ for the decay of M_i .

$$x' = -\sigma x, \quad y' = -\mu y + \frac{1}{2}\sigma x, \quad z' = \frac{1}{2}\sigma x + \mu y$$

$$\sigma = \ln T_s; \quad \mu = \ln T_M$$

c) Solve these equations for x, y & z

$$x(0)=1 \therefore \underline{x = e^{-\sigma t}}$$

$$y' + \mu y = \frac{1}{2}\sigma e^{-\sigma t} \leftarrow \text{needs integrating factor} \rightarrow e^{\int \mu(t) dt}$$

w/ $P(t)$ from $y' + P(t)y = Q(t)$

$\therefore$ int factor is $e^{\mu t}$

$$e^{\mu t} (y' + \mu y) = \frac{1}{2}\sigma e^{(\mu t - \sigma t)}$$

$$\frac{d}{dt} e^{\mu t} \cdot y = \frac{1}{2}\sigma e^{(\mu t - \sigma t)} \leftarrow \text{integrate both sides}$$

$$y e^{\mu t} = \frac{1}{2}\sigma / (\mu - \sigma) e^{(\mu - \sigma)t} + C$$

$$y = \frac{\sigma}{2(\mu - \sigma)} e^{-\sigma t} + C / e^{\mu t} \leftarrow y(0)=0$$

$$\text{so } C = \frac{-\sigma}{2(\mu - \sigma)} \Rightarrow \underline{y = \frac{\sigma}{2(\mu - \sigma)} (e^{-\sigma t} - e^{-\mu t})}$$

$$\dot{z} = 1 - x - y \quad ;)$$

d) At what time does the quantity of M_i peak?

$$\mu y = \frac{1}{2}\sigma e^{-\sigma t} \leftarrow \text{critical point}$$

$$\therefore \frac{\mu \sigma}{2(\mu - \sigma)} (e^{-\sigma t} - e^{-\mu t}) = \frac{1}{2}\sigma e^{-\sigma t}$$

$$\mu \cdot e^{-\sigma t} (e^{-\sigma t} - e^{-\mu t}) = e^{-\sigma t}$$

$$\mu (e^{-\sigma t} - e^{-\mu t}) = (\mu - \sigma) (e^{-\sigma t}) \Rightarrow \mu e^{-\mu t} = \sigma e^{-\sigma t}$$

$$\ln \mu - \mu t = \ln \sigma - \sigma t \Rightarrow t(-\sigma - \mu) = \ln \sigma - \ln \mu$$

$$\therefore t = \frac{\ln \sigma - \ln \mu}{\sigma - \mu}$$

4 cont) e) suppose that instead of $x(0)=1$, we had

$x(0)=2$. What change would that make to $x, y, \dot{x}$?

By linearity everything would double

f) suppose that $x(t)=e^t$ is a solution to the differential equation $x' + 2x = q(t)$? What is the general solution?

$$q(t) = te^t + 2e^t = e^t(t+2)$$

$$\text{take } x' + 2x = 0 \Rightarrow t \frac{dx}{dt} + 2x = 0 \Rightarrow dx/x = -2 \frac{dt}{t}$$

$$\ln|x| = -2 \ln|t| + C$$

$$\Rightarrow x = C/t^2$$

$$\therefore \text{general solution} = e^t + C/t^2$$

5) a) Make a table with three columns. Each row will contain three representations of a complex number z :

the "rectangular" expression $z = a+bi$, the polar expression $|z|$, $\text{Arg}(z)$, & a picture of the complex plane with the complex number marked on it

#	$a-bi$	$ z \arg(z)$	graph
(i)	$1-i$	$\sqrt{2} \arg(-\pi/4)$	
(ii)	$2+i\sqrt{3}$	$ z \arg(\pi/6)$	
(iii)	$(-1-i)/\sqrt{2}$	$ z \arg(5\pi/4)$	
(iv)	$(1+\sqrt{3}i)/\sqrt{2}$	$ z \arg(\pi/3)$	
(v)	$(-1+i)/\sqrt{2}$	$ z \arg(3\pi/4)$	

5 cont) b) Find the complex roots of the following equations: i) $z^4 + 4 = 0$ ii) $z^2 + 2z + 2$

i) $\pm 1 \pm i$

ii) $-1 \pm i$

6) a) Add a fourth column to table you made in 5(c) by giving the exponential representation $z = A e^{i\theta}$

#	$a - bi$	$ z \arg(z)$	$A e^{i\theta}$
(i)	$1 - i$	$\sqrt{2} \arg(-\frac{\pi}{4})$	$\sqrt{2} e^{-i\pi/4}$
(ii)	$2 + i \sin(\frac{\pi}{6})$	$ z \arg(\pi/6)$	$2 e^{i\pi/6}$
(iii)	$(-1 - i)/\sqrt{2}$	$ z \arg(\pi/4)$	$e^{i5\pi/4}$
(iv)	$(1 + \sqrt{3}i)/\sqrt{2}$	$ z \arg(\pi/3)$	$e^{i\pi/3}$
(v)	$(-1 + i)/\sqrt{2}$	$ z \arg(3\pi/4)$	$e^{i3\pi/4}$

b) Find all complex numbers $z = a + bi$ such that $e^z = -2$
 $e^a e^{bi} = -2 \Rightarrow a = \ln 2; b = \pi + 2\pi n \quad n \in \mathbb{N}$

c) Find an expression for $\cos(4t)$ in terms of sums of powers of $\sin t$ & $\cos t$ by using $(e^{it})^4 = e^{4it}$ & Euler's formula.

Use $\Re(e^{4it}) \Rightarrow \cos(4t) = \Re((\cos t + i \sin t)^4)$

$\hookrightarrow$ this is messy & I don't really feel like doing it

7) a) Express $\operatorname{Re}\left(\frac{e^{3it}}{\sqrt{3}+i}\right)$ in the form $a \cos(3t) + b \sin(3t)$. Then rewrite this in the form $A \cos(3t - \phi)$.

$$\frac{\sqrt{3}-i}{4} (\cos(3t) + i \sin(3t)) \rightarrow \frac{\sqrt{3}}{4} \cos(3t) + \frac{1}{4} \sin(3t)$$

$$A = 1/2, \phi = \pi/6$$

b) Find a solution of the differential equation

$z' + 3z = e^{2it}$ of the form $w e^{2it}$, where w is some complex number.

$$w 2i e^{2it} + 3w e^{2it} = e^{2it}$$

$$w(2i + 3) = 1 \Rightarrow w = 1/(2i + 3) \Rightarrow w = \frac{3-2i}{13}$$

$$\therefore \text{solution is } \frac{3-2i}{13} e^{2it} + c e^{-3t}$$

c) Finally, find a solution of $x' + 3x = \cos(2t)$ by relating this ODE to the one in (b). What is the general solution?

This is just the real part of the ODE from part (b) $\therefore$ solution is $x' = (3 \cos(2t) + 2 \sin(2t)) / 13 + c e^{-3t}$

LECTURE 9: Linear 2nd order ODEs with constant coeffs

standard form: $y'' + A y' + B y = 0$ _{homogeneous}
 hence second order

in general, A & B might be functions which vary with independent variables, and the right hand might as well. But we'll look at those later.

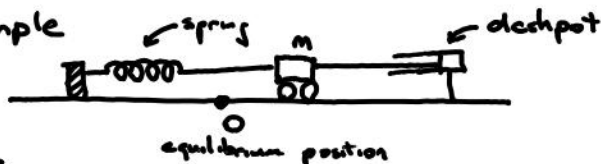
initial conditions satisfied by two constants for 2nd order

Assume general solution: $y = c_1 y_1 + c_2 y_2$ where y_1, y_2 are solutions

$\therefore$ task is reduced to somehow finding

two solutions y_1 & y_2 . where do we get these from?

Standard example



force $\downarrow$ spring $\downarrow$

$$m x'' = -kx - c x' \quad \text{dashpot resists as function of velocity}$$

so $x'' + c/m x' + k/m x = 0$ in standard form

To solve ODE, find two independent solutions (i.e. y_2 not linear combo of y_1)

Basic method: try $y = e^{rt}$ ind. variable

$$r^2 e^{rt} + A r e^{rt} + B e^{rt} = 0; \text{ choose } r \text{ to get } 0$$

$r^2 + Ar + B = 0$ characteristic equation of the system

Three cases!

1) Roots real & not equal. Gen solution $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

Ex $y'' + 4y' + 3y = 0$

$$r^2 + 4r + 3 = 0 = (r+3)(r+1)$$

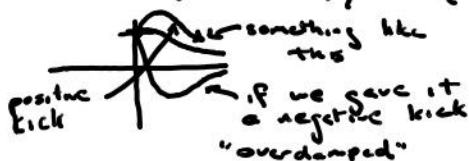
$\therefore$ gen solution $y = c_1 e^{-3t} + c_2 e^{-t}$

suppose $y(0) = 1$ & $y'(0) = 0$ (initial conditions)

$$y' = -3c_1 e^{-3t} - c_2 e^{-t}$$

$$0 = -3c_1 - c_2 \quad \& \quad 1 = c_1 + c_2 \Rightarrow 1 = c_1 - 3c_1 = -1/2 = c_1, \quad 3/2 = c_2$$

solution: $y = -\frac{1}{2} e^{-3t} + \frac{3}{2} e^{-t}$



2) complex roots $r = a \pm bi$ ← i.e. roots are complex conjugates

We get a complex solution $y = e^{t(a+bi)}$

If we have complex solution $u+iv$ to a real differential equation $y'' + Ay' + By = 0$ then u & v are real solutions.

Proof: $(u+iv)'' + (u+iv)'A + (u+iv)B = 0$

$$\underbrace{u'' + Au' + Bu}_{\text{real part}} + i \underbrace{(v'' + Av' + Bv)}_{\text{imaginary part}} = 0$$

The only way for this $= 0$ is if real & complex components are both zero!

$\therefore$ real part & imaginary part of $u+iv$ are both solutions of the ODE (only applies if A & B are reals; otherwise we can't tell what's the real part vs. imaginary part)

solution: $y = e^{at+ibt} \rightarrow Re^{at}(\cos bt)$

$$II e^{at}(\sin bt)$$

$$y = e^{at}(c_1 \cos bt + c_2 \sin bt) \leftarrow \text{sinusoidal}$$

try $y'' + 4y' + 5y = 0$ ↑
stiffer spring

$$r^2 + 4r + 5 = 0; r = \frac{-4 \pm \sqrt{4-20}}{2} = -2 \pm i$$

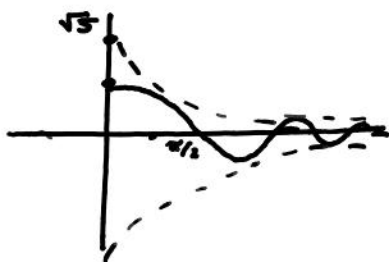
$$y = e^{(2+i)t} = e^{-2t}(c_1 \cos t + c_2 \sin t)$$

try $y(0) = 1$ & $y'(0) = 1 \leftarrow$ solve as before

$$y = e^{-2t}(\cos t + 2 \sin t)$$

$$y = e^{-2t} (\cos t + 2 \sin t) \text{ looks like ...}$$

$$\sqrt{5} e^{-2t} \cos(t - \phi)$$



"underdamped" case

3) Critically damped: two equal roots

$$r = -a \text{ (the root)}$$

$$(r + a)^2 = 0 = r^2 + 2ra + a^2$$

characteristic eqn

$$\text{ODE: } y'' + 2ay' + a^2 y = 0$$

Solution: $y = e^{-at}$ where do we get the other solution from? if you know one

solution to $y'' + py' + qy$ then

$$2a \cdot y' = -a^2 e^{-at} u + e^{-at} u' \quad y = y_1 u \text{ is another solution.}$$

$$1 \cdot y'' = a^2 e^{-at} u - 2ate^{-at} u' + e^{-at} u''$$

combining in this way lets us add and cancel. then

$$e^{-at} u'' \text{ is all that's left. } e^{-at} u'' = 0; u'' = 0$$

$$u = c_1 t + c_2$$

just t is enough!

LECTURE 10: Continuation: Complex Characteristic Root

Oscillations: $y'' + by' + ky = 0$

try $y = e^{rt}$

characteristic equation: $r^2 + br + k = 0$; oscillations correspond to complex roots of the characteristic equation. $r = a + bi$.

$r = a + bi \rightarrow e^{(a+bi)t}$; separate into real & imaginary parts to get $y_1 = e^{at} \cos bt$ (real) & $y_2 = e^{at} \sin bt$ (imaginary). y_1 & y_2 are both real solutions. $a - bi$ would give $y_1 = e^{at} \cos bt$ & $y_2 = -e^{at} \sin bt$, which is just a linear combination of the roots we get from $a + bi$.

Other approach: $y = C_1 e^{(a+bi)t} + C_2 e^{(a-bi)t}$ as a general solution

But this is a complex solution, so we have to get the real solutions out when C_1 & C_2 complex etc. If we split $C_1 = c_1 + i \cdot d_1$ etc and multiply out we can solve but it takes forever. "Hack method"

$u + iv \rightarrow v = 0$
 want real $\rightarrow$ change i to $-i$ & see if expression changes; will give conditions on C_1 & C_2

① $C_1 e^{(a+bi)t} + C_2 e^{(a-bi)t}$

$i \rightarrow -i$

② $\bar{C}_1 e^{(a-bi)t} + \bar{C}_2 e^{(a+bi)t}$

at point ① & ② are the same we have real solutions!

so $C_2 = \bar{C}_1$ and $C_1 = \bar{C}_2$

$\therefore$ real solution: $(c + i \cdot d) e^{(a+bi)t} + (c - i \cdot d) e^{(a-bi)t}$

many scientists & engineers write it this way

Convert $(c+id)e^{(a+bi)t} + (c-id)e^{(a-bi)t}$ to sines & cosines. Again, there's a hack method that takes forever & a better method.

Nicer: pull out $e^{at} \Rightarrow e^{at} [c(e^{ibt} + e^{-ibt}) + id(e^{ibt} - e^{-ibt})]$

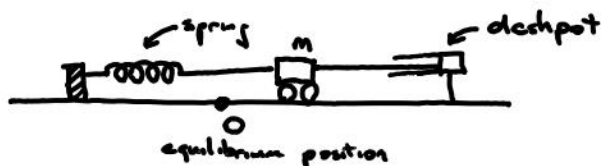
Remember: $\cos a = (e^{ia} + e^{-ia})/2$

$\sin a = (e^{ia} - e^{-ia})/2i$

$\therefore (e^{ibt} + e^{-ibt}) = 2\cos bt$ & $(e^{ibt} - e^{-ibt}) = 2i\sin bt$

so whole expression is $e^{at}(2c\cos bt - 2d\sin bt)$

Dashpot again:



$$m x'' + c x' + k x = 0$$

$$x'' + c/m x' + k/m x = 0$$

$y'' + 2p y' + \omega_0^2 y = 0$ ← notation change for better formulas et cetera

$$r^2 + 2pr + \omega_0^2 = 0$$

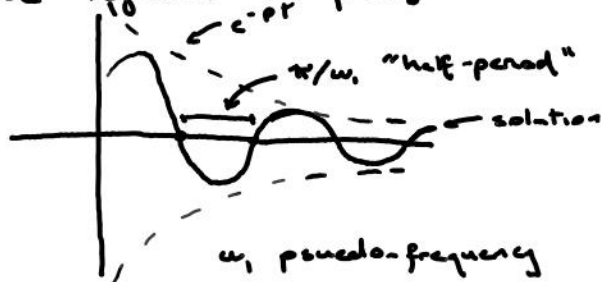
$$r = -p \pm \sqrt{p^2 - \omega_0^2}$$

if $p = 0$, undamped; $y'' + \omega_0^2 y = 0$; $r = \pm i\omega_0$

$$y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t \text{ or } y = A \cos(\omega_0 t - \phi)$$

else damped case we get oscillations if roots are complex;

$p^2 - \omega_0^2$ must be negative. i.e. $p < \omega_0$

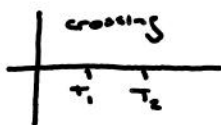


if $c \uparrow$, $\omega_1 \downarrow$

$$r = -p \pm \sqrt{-(\omega_1 p)^2} = -p \pm i\sqrt{\omega_1^2 c}$$

solution: $e^{-pt} (c_1 \cos \omega_1 t + c_2 \sin \omega_1 t)$ or $e^{-pt} A \cos(\omega_1 t - \phi)$

ω_1 is still pseudofrequency



$$t_2 = 2\pi/\omega_1 + t_1; \quad \omega_1 t - \phi = \pi/2$$

Dependencies

p - only on ODE

ϕ
 $\wedge$ } depend on initial conditions

ω_1 - only on ODE ($\omega_0^2 - p^2 = \omega_1^2$) $\triangle_{\omega_1}^{\omega_0, p}$

Problem Set #3

b) skipping - mathlet required

- a) Working with the equation $\dot{y} = (1-y)y - a$ with $a = 3/16$, let y_0 be the stable critical point. Write $u = y - y_0$ for the population excess over equilibrium.
- c) Rewrite the differential equation as a differential equation for u . Check that the new equation is again autonomous and that $u=0$ is a critical point for it.

$$\dot{u} = (1-(u+y_0))(u+y_0) - a: \text{ no other variable, autonomous}$$

9 cont) b) For small u we can neglect higher powers of u (like u^2). This process is "linearisation near equilibrium". What is the linearised equation near $u=0$? What is the general solution to this equation?

$$\dot{u} = (1 - u + y_0)(u - y_0) - a$$

$$= u - u^2 - uy_0 - u^2 + uy_0 - a \Rightarrow u - \cancel{2u^2} - a = u - a$$

General solution to $\dot{u} = u - a = a + Ce^+$

c) At least in this case, where solutions of the original autonomous equation get near equilibrium, they are well-modelled by solutions of the linearisation. Give an approximation of y near equilibrium. Use it to answer this question:

if $y(10) - y_0 = b$, estimate $y(11)$ & $y(12)$

$y \approx a + ce^{+t/2} \Rightarrow y(1) \text{ \& } y(2)$ are exponential decays off $y(10)$

d) Suppose that the linear equation $\dot{x} = p(t)x = q(t)$ is autonomous. What can you say about $p(t)$ & $q(t)$? $p(t)$ & $q(t)$ cannot actually vary with non- x terms - only constants are allowed for equation to be autonomous

10-12) 10 missing; 11 & 12 use applets

LECTURE 11: Theory of General Second-Order Linear Homogeneous ODEs

linear in $y, y', y'' \rightarrow y'' + p(x)y' + q(x)y = 0$

solution method for 2nd-order ODEs: find y_1 & y_2

independent solutions; i.e. $y_2 \neq c y_1$, $\dot{y}_1 \neq c \dot{y}_2$

Then all solutions are:

because of the $y_1=0$ case

$$y = c_1 y_1 + c_2 y_2. \text{ WHY?}$$

Q1: Why are all $c_1 y_1 + c_2 y_2$ solutions?

Q2: Why does this cover all solutions?

Q1: Superposition principle:

THEY'RE
CONSTRUCTING
A VECTOR SPACE!
 y_1, y_2 are
bases!

y_1, y_2 are solutions to linear homogeneous eq.

$\Rightarrow c_1 y_1 + c_2 y_2$ is a solution (linear combination)

$$y'' - p y' + q y = 0$$

$$D^2 y - p D y + q y = 0 \Rightarrow \underbrace{(D^2 - p D + q)}_{\text{"L"} \Rightarrow L y = 0} y = 0$$

not a multiplication; an application

Think of L as a black box: $u(x) \rightarrow \boxed{L} \rightarrow v(x)$

$$L(u_1 + u_2) = L(u_1) + L(u_2) \quad \text{I.e. } L \text{ is a linear operator}$$

$$L(cu) = cL(u) \quad [D \text{ is also linear!}]$$

Proof of superposition principle: ODE: $Ly = 0$

$$L(c_1 y_1 + c_2 y_2) = 0 = L(c_1 y_1) + L(c_2 y_2) = c_1 L(y_1) + c_2 L(y_2)$$

$$= c_1 \cdot 0 + c_2 \cdot 0 \quad \checkmark$$

Q2: why are these all the solutions? solution is a 2D vector space spanned by y_1, y_2

Detour: Solving the initial value problem.

$\{c_1 y_1 + c_2 y_2\}$ is enough to satisfy any initial value conditions

Proof: $y(x_0) = a$ $y = c_1 y_1 + c_2 y_2$ $c_1 y_1(x_0) + c_2 y_2(x_0) = a$
 $y'(x_0) = b$ $y' = c_1 y_1' + c_2 y_2'$ etc for b
 plug in $x = x_0$

$$M = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \text{ non-singular } \Rightarrow \text{solvable}$$

now we have a pair of linear eqs. w/ c_1, c_2 variables solvable as long as y_1, y_2 independent

$|M|$ is the "Wronskian" $\Rightarrow W(y_1, y_2)$

a function of x , because y_1, y_2 are functions of x

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Theorem: if y_1, y_2 are solutions to ODE, $W(y_1, y_2) \neq 0$ or $W(y_1, y_2)$ is never 0 for any value of x .

$\{c_1 y_1 + c_2 y_2\} = \{c_1' u_1 + c_2' u_2\}$ where u_1, u_2 are any other pair of ind. solutions

Finding normalised solutions. Isn't this Gram-Schmidt?

at zero: $\bar{y}_1, \bar{y}_2$ satisfy special initial conditions

$$\bar{y}_1(0)=1; \bar{y}_1'(0)=0$$

$$\bar{y}_2(0)=0; \bar{y}_2'(0)=1$$

Ex: $y'' + y = 0$

$$y_1 = \cos x, y_2 = \sin x$$

orthonormal solutions!

Ex: $y'' - y = 0$

$$y_1 = e^x; y_2 = e^{-x}$$

gen solution: $y = c_1 e^x + c_2 e^{-x}$

get $\bar{y}_1, \bar{y}_2$: $y' = c_1 e^x - c_2 e^{-x}$

$$\therefore \bar{y}_1: c_1 + c_2 = 1$$

$$c_1 - c_2 = 0$$

$$\therefore c_1 = 1/2 = c_2 \Rightarrow \bar{y}_1 = (e^x + e^{-x})/2 \quad [\bar{y}_2 = \sinh x]$$

Normalised solutions are easier to work with! note

$\bar{y}_1$ is $\cosh x$ & $\bar{y}_2$ is $\sinh x$. if $\bar{y}_1, \bar{y}_2$ normalised at 0,

solution to IVP ODE + $\begin{cases} y(0) = y_0 \\ y'(0) = y'_0 \end{cases}$

$$= y_0 \bar{y}_1 + y'_0 \bar{y}_2 \leftarrow \text{normalising solutions instantly solves any set of i.c.s.}$$

Q2 ANSWERED: Existence & uniqueness theorem!

$$y'' + p y' + q y = 0 \leftarrow \text{assume } p, q \text{ cont for all } x$$

There is one & only one soln s.t. $y(0) = A$ & $y'(0) = B$

If we want all solutions to ODE, claim $C_1 \bar{Y}_1 + C_2 \bar{Y}_2$ contains all solutions. Because given solution $u(x)$, $u(x_0) = u_0$, $u'(x_0) = u'_0 \Rightarrow u_0 \bar{Y}_1 + u'_0 \bar{Y}_2$ satisfies u.c.s.

LECTURE 12: General Theory for Inhomogeneous ODEs

$y'' - p(x)y' + q(x)y = f(x) \leftarrow$ up until now this has been zero for homogeneous equations.
 mostly x will be time. $f(x)$ is input/signal/forcing term

solution is the "response" to the input. we cannot get solutions without understanding the solution to the equivalent homogeneous equation $y'' + p(x)y' + q(x)y = 0$
 $y_h = C_1 y_1 + C_2 y_2$ (homogeneous / complementary soln)

Ex: $\underbrace{mx'' + bx' + kx = f(t)}_{\text{forced system}} \leftarrow$ spring-mass-dashpot system, with $f(t)$ as external force, homogeneous case is "passive" system

Ex:  $L i' + R i + \frac{1}{C} \int i dt = E(t); \quad q' = i$
 $L i i' + R i^2 + \frac{1}{2C} i^2 = E'(t)$
 $\hookrightarrow$ forced

Theorem: $Ly = f(x)$ L 2nd order linear operator

Solution: $y_p + y_h$ ^{← noting} i.e. $y_p = c_1 y_1 + c_2 y_2$

y_p is a particular solution

1) find y_h

2) find y_p : any one solution

need to prove that $y_p + c_1 y_1 + c_2 y_2$ are solutions

$$\text{proof: } L(y_p + c_1 y_1 + c_2 y_2) \Rightarrow L(y_p) + \underbrace{L(c_1 y_1) + L(c_2 y_2)}_0$$

$\downarrow$
 $f(x)$; since y_p
is solution

$$\therefore \text{ satisfy } L(y) = f(x)$$

need to prove there are no other solutions

$u(x)$ is a solution, so $L(u) = f(x)$

$$L(y_p) = f(x)$$

$$L(u - y_p) = 0$$

$$\therefore u - y_p = \hat{c}_1 y_1 + \hat{c}_2 y_2$$

$\therefore u$ in solution space

So to solve inhomogeneous equations, we just need to find y_p . (this is hard though!) Let's see if we can relate this to first order equations with constant

coefficients. $y' + ky = q(t) \Rightarrow y = \underbrace{e^{-kt} \int q(t) e^{kt} dt}_{y_p} + \underbrace{c e^{-kt}}_{y_h}$

$k > 0$ $y_h \rightarrow 0$ as $t \rightarrow \infty$, transient; s.s. y_p

$k < 0$ y_h dominates as $t \rightarrow \infty$ so no transients or steady state solutions. How do we relate this to higher-order ODEs?

$$y'' + Ay' + By = f(t)$$

$y = y_p + y_h \leftarrow y_h$ contains initial conditions

can we make $c_1 y_1 + c_2 y_2$ go to 0 as $t \rightarrow \infty$?

if this is so, the ODE is called "asymptotically stable"
and we have a steady-state & a transient again.

How do we tell if an ODE is stable?

chara. roots	solutions	stability condition
$r_1 \neq r_2$	$c_1 e^{r_1 t} + c_2 e^{r_2 t}$	$r_1 \& r_2$ negative
$r_1 = r_2$	$(c_1 + c_2 t) e^{r t}$	r negative
$r = a \pm bi$	$e^{at} (c_1 \cos bt + c_2 \sin bt)$	$A > 0$

An ODE is stable if all the characteristic roots have negative real part

LECTURE 13: Finding Particular Solutions to Inhomogeneous Equations

$y'' + Ay' + By = f(x)$; find a part solution y_p ; gen solⁿ $y_p + y_h$
important r.h. sides: e^{ax} (a often neg), $\sin/\cos wx$, $e^{ax} \sin/\cos wx$
special cases of $e^{(a+iu)x} / e^{ax}$ [x complex]

$$y'' + Ay' + By = f(x) \rightarrow \underbrace{(D^2 + AD + B)}_{P(D)} y = f(x)$$

Just swap a for D!
 $P(D) e^{ax} = P(a) e^{ax}$ - obvious by linearity & differentiation
 $= D^2 e^{ax} + AD e^{ax} + B e^{ax}$ for $D \hat{=} D^2 \therefore$ also true for $P(D)$

Exponential Input Theorem

for $y'' + Ay' + B = e^{\alpha x} \rightarrow y' = e^{\alpha x} / p(\alpha); p(\alpha) \neq 0$

Proof: $p(D)y_p = e^{\alpha x}$

$$p(D) e^{\alpha x} / p(\alpha) = \frac{p(\alpha) e^{\alpha x}}{p(\alpha)} = e^{\alpha x}$$

Ex: $y'' - y' + 2y = 10e^{-x} \sin x$

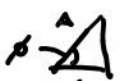
find particular solution: $(D^2 - D + 2)\tilde{y} = 10e^{(-1+i)x}$ complexified

$$\tilde{y}_p = 10e^{(-1+i)x} / [(-1+i)^2 - (-1+i) + 2]$$

since $10e^{-x} \sin x$ is imaginary part of $10e^{(-1+i)x}$, we need

imaginary part of $\tilde{y}_p$ ← real ← imaginary

$$\tilde{y}_p = 10e^{(-1+i)x} / ([(-1-1) + 1 + 2] + [-i - i - i]) = \frac{10}{3} \frac{(1+i)}{2} e^{-x} (\cos x + i \sin x)$$

$$y_p = \text{Im}(\tilde{y}_p) = \frac{5}{3} e^{-x} (\cos x + \sin x)$$
$$= \frac{5}{3} e^{-x} \sqrt{2} \cos(x - \pi/4)$$


what happens when $p(\alpha)$ is zero? Exponential Shift Rule

[we're going to call " α " " a " from now on.]

$$p(D)e^{ax}u(x) = e^{ax}p(D+a)u(x)$$

Proof: special case? $p(D)=D \Rightarrow D e^{ax}u = e^{ax}Du + a e^{ax}u$

$$= e^{ax}(D+a)u$$

Now if it also works for D^2 always true by l.i.n.

$$p(D)=D^2 \Rightarrow D^2 e^{ax}u = D(D e^{ax}u) = D(e^{ax}(D+a)u)$$
$$= e^{ax}(D+a)[(D+a)u]$$
$$= e^{ax}(D+a)^2 u$$

(generalises to higher-order by induction)

So if $P(a) = 0$, $(D^2 + AD + B)y = e^{ax}$

$y_p = x e^{ax} / P'(a)$. if a is a simple root, $P'(a) \neq 0$

if a is a double root ...

$y_p = x^2 e^{ax} / P''(a)$. for higher-order keep going :)

Proof - simple root case

$$P(D) = (D-b)(D-a) \text{ with } b \neq a$$

$$P'(D) = D-a + D-b \Rightarrow P'(a) = a-b$$

$$\text{Try } P(D) e^{ax} \cdot x / P'(a) \stackrel{?}{=} e^{ax}$$

$$e^{ax} (D+a-b) \tilde{x} / P'(a) = e^{ax} = e^{ax} \frac{(a/b)}{(a-b)} \quad \checkmark$$

Ex. $y'' - 3y' + 2y = e^x$ 1 is a simple root of $D^2 - 3D + 2$

$$y_p = x e^x / P'(1) = -x e^x$$

LECTURE 14: Interpretation of Exceptional Case - Resonance

$P(D)$: formal polynomial in D is a linear operator on functions like $y(t)$

we need to be comfortable w/ dual role!

$f(t) \sim \text{exponential(ish)} e^{at}$

$$y_p = e^{at} / P(a) \Rightarrow t e^{at} / P'(a) \Rightarrow t^2 e^{at} / P''(a) \Rightarrow \text{etc.}$$

Resonance: what does this have to do with ODEs?

$$y'' + \omega_0^2 y = \cos \omega_1 t \quad \leftarrow \text{forcing term, } \omega_1 \neq \omega_0$$

nat. freq. $\nearrow$

$$(D^2 + \omega_0^2) y = \cos \omega_1 t \Rightarrow (D^2 + \omega_0^2) \tilde{y} = e^{i\omega_1 t} \Rightarrow \tilde{y}_p = \frac{e^{i\omega_1 t}}{(\omega_0^2 - \omega_1^2)}$$

complex f.g.

$$\therefore y_p = \cos \omega_1 t / (\omega_0^2 - \omega_1^2) \quad \leftarrow \omega_0^2 - \omega_1^2 \quad \omega_0^2 - \omega_1^2$$

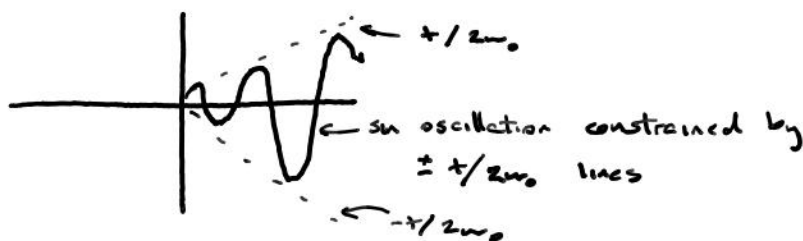
Big amplitude response if driving frequency $\sim$ natural frequency

For $\omega_0 = \omega_1$; $(D^2 + \omega_0^2)y = \cos \omega_0 t$

$$\hookrightarrow (D^2 + \omega_0^2)\tilde{y} = e^{i\omega_0 t} \quad i\omega_0 \text{ is a simple zero of } D^2 + \omega_0^2$$

$$\therefore \tilde{y}_p = t e^{i\omega_0 t} / 2i\omega_0 \quad \leftarrow 2D$$

$$\text{Re}(\tilde{y}_p) = t \sin(\omega_0 t) / 2\omega_0 = y_p$$



Remember: any solution is a particular solution!

$$(D^2 + \omega_0^2)y = \cos \omega_1 t \Rightarrow y_p = \frac{\cos \omega_1 t}{\omega_0^2 - \omega_1^2} - \frac{\cos \omega_0 t}{\omega_0^2 - \omega_1^2}$$

linear combo of y_h is valid!

$$\lim_{\omega_1 \rightarrow \omega_0} \frac{\cos \omega_1 t - \cos \omega_0 t}{\omega_0^2 - \omega_1^2} = \lim_{\omega_1 \rightarrow \omega_0} \frac{-\sin(\omega_1 t) \cdot t}{-2\omega_1} = \textcircled{1}!$$

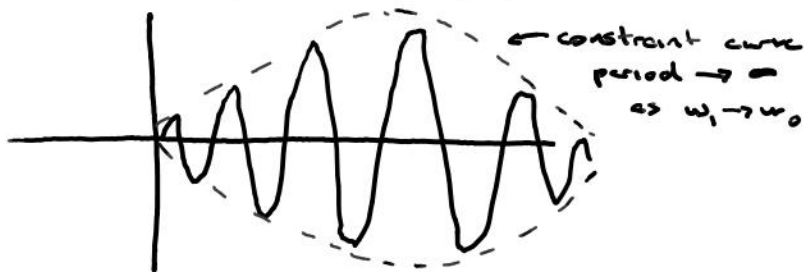
$\nwarrow$
L'Hopital

Geometric meaning? $\cos b - \cos a = 2 \sin\left(\frac{A-B}{2}\right) \sin\left(\frac{A+B}{2}\right)$

$$b = \omega_1 t \quad a = \omega_0 t$$

$$\therefore \frac{\cos \omega_1 t - \cos \omega_0 t}{\omega_0^2 - \omega_1^2} = \frac{2 \sin\left(\frac{\omega_0 - \omega_1}{2} t\right) \sin\left(\frac{\omega_0 + \omega_1}{2} t\right)}{\omega_0^2 - \omega_1^2}$$

$\underbrace{\hspace{10em}}_{\text{varying amplitude } \sin \omega_0}$



Damped Resonance

$x'' + px' + w_0^2 x = f(t)$; w_0^2 = nat. freq. of undamped system

driving freq ω

w_1 : pseudofrequency (page 38)

$$x'' + 2px' + w_0^2 x = \cos \omega t$$



$$w_1^2 = w_0^2 - p^2 \quad \left(\begin{array}{l} \text{from char} \\ \text{roots of} \\ \text{damped eqn} \end{array} \right)$$

prob: which ω gives max amplitude of response?

$$\text{ans: } w_r = \sqrt{w_0^2 - 2p^2}$$

LECTURE 15: FOURIER SERIES (INTRODUCTION)

So far we've been working with $y'' + ay' + by = f(t)$ with soln $y(t)$ the response to the input $f(t)$ - but that input has always been exponential. Are we only dealing with special cases? Fourier Series show that any $f(t)$ which is periodic & not too discontinuous can be represented as an infinite sum of sines & cos.

$$f(t) = c + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt \quad \text{where } n \text{ is frequency}$$

Input

$$\sin(nt)$$

$$\cos(nt)$$

$$\sin(nt) + \cos(nt)$$

$$a_n \cos(nt)$$

$$f(t)$$

Response

$$y_n^{(1)}(t)$$

$$y_n^{(2)}(t)$$

$$y_n^{(1)}(t) + y_n^{(2)}(t)$$

$$a_n y_n^{(1)}(t)$$

$$\sum_{n=1}^{\infty} a_n y_n^{(1)}(t) + b_n y_n^{(2)}(t) + c_1$$

superposition principle;
ODE is linear

$\therefore$ Fourier gives us response to any periodic function w/ period 2π

Today we'll learn to compute Fourier series for periodic functions.

$u(t), v(t)$ functions on $\mathbb{R}$

(say 2π is a period)

These functions are orthogonal on $[-\pi, \pi]$ if $\int_{-\pi}^{\pi} u(t) \cdot v(t) dt = 0$

Theorem $\begin{cases} \sin nt & 1 \rightarrow \infty & u(t) \\ \cos mt & 0 \rightarrow \infty & v(t) \end{cases} \quad \left[\int_{-\pi}^{\pi} \sin^2 nt dt = \int_{-\pi}^{\pi} \cos^2 nt dt = \pi \right]$
 doesn't work if $u=v$

any two distinct instances of these functions are orthogonal on $[-\pi, \pi]$.

Why is this? Many ways to prove. Trig identities, complex exponentials ... or use the ODE. This method is general for orthogonal functions rather than 'just sines and cosines'.

Assume $m \neq n$; $\sin nt, \cos nt$ satisfy $u'' + n^2 u = 0 \Rightarrow u'' = -n^2 u$

Let $u_n \neq v_m$ be any two of the functions

$\int_{-\pi}^{\pi} u_n'' v_m dt$
 extremely asymmetric; need to integrate by parts for balancing

$\int_{-\pi}^{\pi} u_n' v_m' dt$
 if either is $\sin$, 0; if $\cos$ $-\pi \rightarrow \pi$ symmetric; $\therefore 0$

since $\int_{-\pi}^{\pi} u_n'' v_m dt = -n^2 \int_{-\pi}^{\pi} u_n v_m dt$
 not symmetric

The proof is that the only way for the symmetry conditions to resolve is by $\int_{-\pi}^{\pi} u_n v_m dt = 0$ for $m \neq n$

What about $m=n$?

$$\int_{-\pi}^{\pi} \cos nt \sin nt \, dt = 0 \quad \checkmark$$

Given $f(t)$ with 2π period we need to find a_n & b_n

$$f(t) = \dots + a_k \cos kt + \dots + a_n \cos nt + \dots$$

$\uparrow$ some other term we want

$$f(t) \cos nt = \dots + a_k \cos kt \cos nt + \dots + a_n \cos^2 nt + \dots$$

$$\int_{-\pi}^{\pi} f(t) \cos nt \, dt = \int_{-\pi}^{\pi} \dots + a_k \cos kt \cos nt + \dots + a_n \cos^2 nt + \dots \, dt$$

this is the only term that survives!
 $= a_n \pi$

$$\therefore a_n \pi = \int_{-\pi}^{\pi} f(t) \cos(nt) \, dt \Rightarrow a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) \, dt \quad n=1, 2, \dots$$

similarly

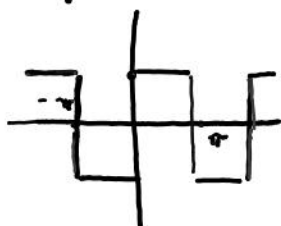
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) \, dt \quad n=1, 2, \dots$$

Don't forget the constant term c_0

$$f(t) = c_0 + \dots + a_n \cos(nt) + \dots$$

$$\int_{-\pi}^{\pi} f(t) \, dt = 2\pi c_0 \Rightarrow c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \, dt$$

Ex: Square Wave



$$a_n = 0$$

split w/c of discontinuity

$$b_n = \int_{-\pi}^0 \sin nt \, dt + \int_0^{\pi} \sin nt \, dt = \frac{1 - \cos n\pi}{n} - \frac{1 - \cos n\pi}{n}$$

$$= \frac{2}{n} (1 - \cos n\pi)$$

$$= \frac{2}{n} (1 - (-1)^n)$$

LECTURE 16: Fourier Series Continued

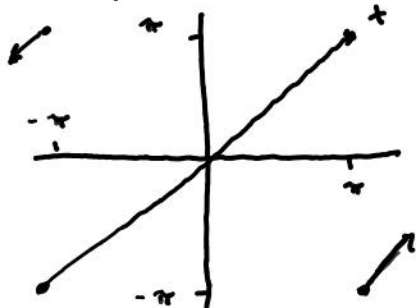
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt \quad \text{where } f(t) \text{ is periodic } \hat{=} \\ \text{2\pi is a period}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt$$

$$f(t) \Rightarrow g(t) \Rightarrow \text{FS for } f = \text{FS for } g$$

(since we have formulae for a_n & b_n)

Calculate Fourier series for:



Calculations are very involved - how do we shorten?

Main trick for simplification: use evenness & oddness:

① $f(t)$ even $\rightarrow$ Fourier series contains only cos terms

$f(t)$ odd $\rightarrow$

sin terms

$$\textcircled{1} f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt$$

show by: $f(-t) = f(t)$

$$\text{FS } f(-t) = a_0/2 + \sum a_n \cos nt - b_n \sin nt$$

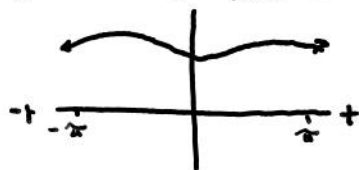
$$= f(t) = a_0/2 + \sum a_n \cos nt + b_n \sin nt$$

stays same when $-t$ subbed for t

$$\text{But } f(-t) = f(t) \hat{=} \text{FS } f(-t) = \text{FS } f(t) \Rightarrow b_n = 0$$

i.e. by uniqueness of Fourier series

if $f(t)$ is even, $f(t)\cos nt$ also even



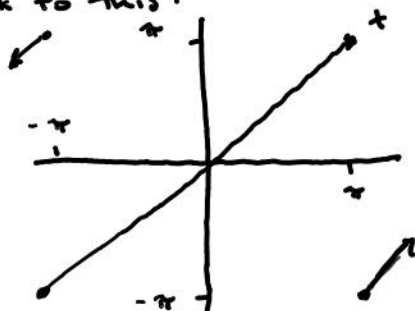
$$\int_{-\pi}^{\pi} f(t) dt \rightarrow 2 \int_0^{\pi} f(t) dt \quad \text{for even functions}$$

so $a_n = \frac{2}{\pi} \int_0^{\pi} f(t) \cos nt dt$ ($b_n=0$) for even functions

$b_n = \frac{2}{\pi} \int_0^{\pi} f(t) \sin nt dt$ ($a_n=0$) for odd functions

this is 'odd' odd function $\Rightarrow$ even!

Back to this:



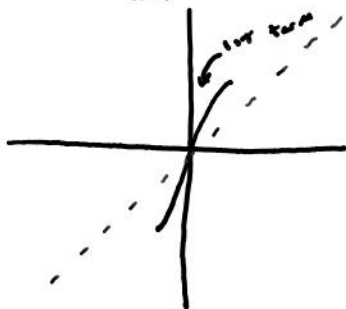
Obviously this is an odd function

so $a_n \rightarrow 0$

$$b_n = \frac{2}{\pi} \int_0^{\pi} t \sin nt dt$$

$$\begin{aligned} \textcircled{1} \quad \frac{2}{\pi} \left[-t \cos nt / n \right]_0^{\pi} - \frac{2}{\pi} \int_0^{\pi} \frac{-\cos nt}{n} dt &= \frac{2}{\pi} \left[\frac{-\pi}{n} (-1)^n + \sin nt / n^2 \Big|_0^{\pi} \right] \\ &= \frac{2}{n} (-1)^{n+1} \end{aligned}$$

$\therefore$ F.S. for $f(t) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt = 2 \left(\sin t - \frac{1}{2} \sin 2t + \frac{1}{3} \sin 3t - \dots \right)$



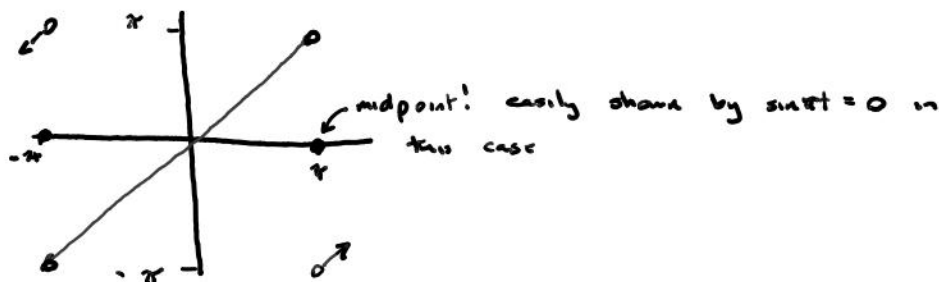
fourier is not trying to approximate functions at 0 - it's over an interval not a point
each successive term corrects over the interval

Theorem: If f is continuous at t_0 , then $f(t)$ is equal to the sum of the Fourier series at t_0

if f is not continuous at t_0 , it isn't :)

if f has a jump discontinuity at t_0 , f.s. converges to the midpoint of the jump

$$2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nt = F(t) \text{ looks like ...}$$



Extending Fourier Series

1) Period is $2L$

use natural functions: $\cos n \frac{\pi}{L} t$



$$\sin n \frac{\pi}{L} t \leftarrow 0 @ 0, L, \pm 2L$$

this is a simple rescaling of the t axis to make $L \rightarrow \pi$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n \pi t / L + b_n \sin n \pi t / L$$

$$a_n = \frac{1}{L} \int_{-L}^L f(t) \cos n \pi t / L dt; \quad b_n = \frac{1}{L} \int_{-L}^L f(t) \sin n \pi t / L dt$$

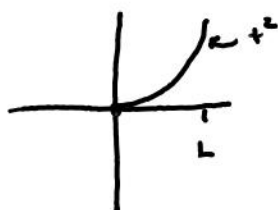
i.e. swap L for π and rescale t by π/L everywhere

2) we've been talking about periodic functions, but another way of considering this is functions with a finite interval of interest. We can just make such a function periodic!

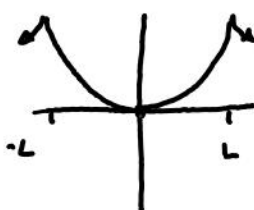
If $f(t)$ defined on $[0, L]$... periodic extension!

Example: t^2

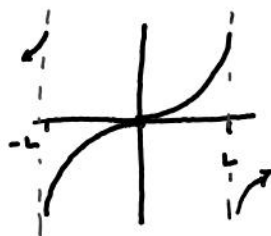
Base



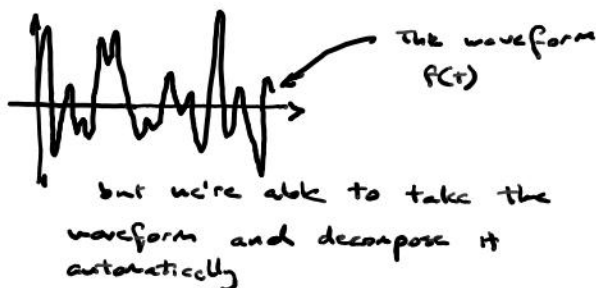
Even



Odd



LECTURE 17:



How does the brain actually achieve hearing the individual notes rather than a mess?

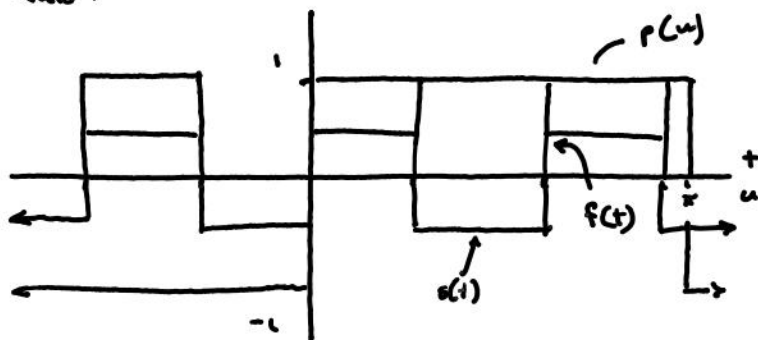


$$\begin{cases} \sin n\pi t \\ \cos n\pi t \end{cases}$$

$L=1$

What is this Fourier series? It's neither even nor odd so last lecture's tricks don't work.

Observe that lowering the wave by $1/2$ makes it an odd function! $f(t) \rightarrow s(t)$ is a vertical shift. In a previous lecture we solved a square wave $p(u)$ which looked like this:



$$p(u) = \frac{4}{\pi} \sum_{\text{odd}} \frac{\sin n\pi u}{n}$$

Now we want to scale the u axis against the t axis to rescale $s(t)$'s period to 2π !

$$f(t) = s(t) + 1/2$$

$$g(u) = \frac{1}{2} p(u)$$

$$\therefore f(t) = \frac{1}{2} \cdot \frac{1}{2} \frac{4}{\pi} \sum_{\text{odd}} \frac{\sin n\pi t}{n}$$

$$x'' + \omega_0^2 x = f(t) \quad \leftarrow \text{find a particular solution } x_p$$

can find x_p if rh is $\cos \omega t$ & $\sin \omega t$ or some multiple

$$x_p = \{\cos \omega t + \sin \omega t\} / (\omega_0^2 - \omega^2)$$

for period $f(t)$
as $2L \rightarrow$

$$\text{if } f(t) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos \omega_n t + b_n \sin \omega_n t \quad \omega_n = n\pi/L$$

then we have the answer by linear superposition!

$$x_p = \frac{a_0}{2\omega_0^2} + \sum_{n=1}^{\infty} \left(\frac{a_n \cos \omega_n t}{\omega_0^2 - \omega_n^2} + \frac{b_n \sin \omega_n t}{\omega_0^2 - \omega_n^2} \right)$$

if input is $f(t)$



$$x'' + \omega_0^2 x = f(t)$$

$$\text{response: } \frac{1}{2\omega_0^2} + \frac{2}{\pi} \sum_{n \text{ odd}} (\sin(n\pi t)) / (\omega_0^2 - (n\pi)^2)$$

Natural frequency $\omega_0 = 10$

$$x_p(t) \approx \frac{1}{200} + \frac{2}{\pi} \left[\sin \pi t / (100 - \pi^2) + \sin 3\pi t / (100 - 9\pi^2) - \sin 5\pi t / (100 - 25\pi^2) \right]$$

$$\approx 0.05 + 0.01 \sin \pi t + 0.015 \sin 3\pi t - \text{something negligible.}$$

i.e. the frequencies that make up the response do not occur with the same amplitudes. near-resonance occurs when $n=3$

frequencies in $f(t)$ are hidden but system picks out $\hat{f}$; resonates with the frequencies closest to its natural frequency. our inner ear vibrates according to $f(t)$ and the cochlea has sensors tuned to natural frequencies. when complex $f(t)$ hits, the sensors respond to signals close to its natural frequency.

Book Method: uses differentiation of f.s. term by term
 so knowing the general undamped response is not required.

Assume solution of form $x_p = c_0 + \sum_1^{\infty} c_n \sin n\pi t$

What are c_n ? Substitute into the ODE!

$$x_p'' = \sum_1^{\infty} -c_n n^2 \pi^2 \sin n\pi t$$

$$\omega_0^2 c_0 + \sum_1^{\infty} c_n (\omega_0^2 - (n\pi)^2) \sin n\pi t = \frac{1}{2} + \sum_{n \text{ odd}} \frac{\sin n\pi t}{n}$$

$$\therefore c_0 = \frac{1}{\omega_0^2} \quad c_n = \frac{2}{\pi n} \frac{1}{\omega_0^2 - (n\pi)^2} \leftarrow \text{same as before}$$

LECTURE 19: The Laplace Transform (nb: LHS variable)

The Laplace Transform comes from a power series

$$\sum_0^{\infty} a_n x^n = A(x); \text{ or } \sum_0^{\infty} a(n) x^n = Ax$$

^ a function on the discrete variable
 $n \in \mathbb{N}$, with $a(n) \in \mathbb{R}$

$$a(n) \xrightarrow{\text{association}} A(x)$$

$$1 \xrightarrow{\text{association}} \frac{1}{1-x}, \quad |x| < 1$$

$$\frac{1}{n!} \xrightarrow{\text{association}} e^x$$

What is the continuous analogue of these power series?

$$n = 0, 1, 2, \dots \rightarrow t \text{ where } 0 \leq t \leq \infty$$

Summation over continuous variables is replaced by integration, so the equivalent to $\sum_0^\infty a(n)x^n = A(x)$ for our continuous variable t is:

$$\int_0^\infty a(t)x^t dt = A(x)$$

this is unpleasant; x^t is not fun to integrate.

but! $x = e^{hx}$ & $x^t = (e^{hx})^t$

Note that this integral cannot converge if $x < 1$, and x is positive to avoid complex numbers.

$0 < x < 1$, $\ln x < 0$. Let $s = -\ln x$

Now let the integral be (switched to f for functions)

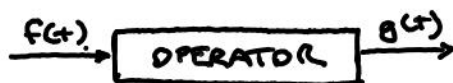
$$\int_0^\infty f(t)e^{-st} dt = F(s)$$

This is the Laplace Transform!

Note that we're using an input $f(t)$ and getting an output $F(s)$ - two different variables. This is because the Laplace Transform is a transform, not an operator!



Fortunately it's a linear transform



$$\mathcal{L}(f(t)) = F(s) \text{ or } f(t) \rightsquigarrow F(s)$$

by linearity, $\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$

$$\mathcal{L}(cf) = c\mathcal{L}(f)$$

(because the integral is a linear operator)

Examples:

$$1 \rightsquigarrow \textcircled{1}$$

$$\textcircled{1}: \int_0^{\infty} 1 \cdot e^{-st} dt = \lim_{R \rightarrow \infty} \int_0^R e^{-st} dt = \lim_{R \rightarrow \infty} \left[\frac{e^{-st}}{-s} \right]_0^R$$

$$= \lim_{R \rightarrow \infty} (e^{-sR} - 1) / -s = 1/s ; s > 0$$

what happens if $s < 0$? this question is meaningless

$$e^{at} \rightsquigarrow \textcircled{2}$$

② Instead of calculating directly, we're going to look at $\mathcal{L}(e^{at} f(t)) = \textcircled{3}$

$$\int_0^{\infty} \underset{\substack{\uparrow \\ \text{kernel}}}{e^{at}} f(t) e^{-st} dt = \int_0^{\infty} f(t) e^{-(s-a)t} dt = F(s-a)$$

for $s > a$, assuming $F(s) s > 0$

③ $e^{at} f(t) \rightsquigarrow F(s-a)$ Exponential shift formula

$$\therefore \textcircled{2} = 1/(s-a)$$

We can extend the result in ③ to complex numbers, so $\mathcal{L}(e^{(a+bi)t}) = 1/(s-(a+bi)) \quad s > 1$

$$\cos at = \frac{e^{iat} + e^{-iat}}{2}$$

$$\begin{aligned} \therefore \mathcal{L}(\cos at) &= \frac{1}{2} \mathcal{L}(e^{iat}) + \frac{1}{2} \mathcal{L}(e^{-iat}) \\ &= \frac{1}{2} \left(\frac{1}{s-ia} + \frac{1}{s+ia} \right) \leftarrow \text{flipping the sign of } i \text{ leaves} \\ &\quad \text{the sum unchanged, } \therefore \underline{\text{real}} \end{aligned}$$

$$= \frac{1}{2} \left(\frac{2s}{s^2 + a^2} \right) \Rightarrow s/(s^2 + a^2); \quad s > 0$$

$$\therefore \mathcal{L}(\sin at) = a/(s^2 + a^2); \quad s > 0$$

Inverting Laplace Transforms is unpleasant and involves dealing with partial fraction decompositions.

$$1/(s(s+3)) \rightsquigarrow ?$$

$$\hookrightarrow \frac{1/3}{s} + \frac{-1/3}{s+3} \rightsquigarrow \frac{1}{3} \cdot 1 - \frac{1}{3} \cdot e^{-3t}$$

$$\mathcal{L}(t^n) = \textcircled{4} \quad \leftarrow \begin{array}{l} \text{integrate by parts, differentiate } t^n \text{ term } \uparrow \\ \text{integrate } e^{-st} \end{array}$$

$$\begin{aligned} &= \int_0^\infty t^n e^{-st} dt \\ &= t^n e^{-st} / (-s) \Big|_0^\infty - \int_0^\infty n t^{n-1} e^{-st} / (-s) dt \end{aligned}$$

($s > 0$)

$$\begin{aligned} &= -\frac{1}{s} \lim_{t \rightarrow \infty} t^n e^{-st} + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} dt = \frac{n}{s} \mathcal{L}(t^{n-1}) \\ &\quad \nearrow \text{by } n \text{ applications of L'Hopital} \end{aligned}$$

$$\begin{aligned} &= \frac{n(n-1)}{s^2} \mathcal{L}(t^{n-2}) \\ &= n! / s^{n+1} = \textcircled{4} \end{aligned}$$